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ABSTRACT

These materials were developed to help high school teachers to become familiar with the approach to tenth-grade Euclidean geometry which was adopted by the School Mathematics Study Group (SMSG). It is emphasized that the materials are unsuitable as a high school textbook. Each document contains material too difficult for most high school students. It is assumed that teachers who study the notes have good backgrounds in axiomatic geometry. In particular, some familiarity with Euclid's Elements is presupposed. Chapters include: (1) Historical Introduction; (2) Logic; (3) Points, Lines, and Planes; (4) Real Numbers and the Ruler Axiom; (5) Separation in Planes and in Space; (6) Angles and the Protractor Postulates; (7) Congruence; (8) Parallelism; (9) Area; and (10) Circles and Spheres. (Author/RH)

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# STUDIES IN MATHEMATICS
# VOLUME II

## *Euclidean Geometry Based on Ruler and Protractor Axioms*

(second revised edition)

By **CHARLES W. CURTIS**, University of Wisconsin
**PAUL H. DAUS**, University of California, Los Angeles
**ROBERT J. WALKER**, Cornell University

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## *Euclidean Geometry Based on Ruler and Protractor Axioms*

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*Written for the* SCHOOL MATHEMATICS STUDY GROUP
*Under a grant from the* NATIONAL SCIENCE FOUNDATION

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# Preface

These notes have been prepared to help high school teachers to become familiar with the approach to tenth grade Euclidean geometry which has been adopted by the School Mathematics Study Group (SMSG). They are intended specifically to be used during the summer of 1959 in courses on geometry for high school teachers. The SMSG is preparing a tenth grade geometry text book and a teachers' manual, and these notes follow the preliminary outline of the text book. It should be emphasized, that these notes are quite unsuitable as a text book for high school students, nor can they be used as a teachers manual. They contain much material which is too difficult to be presented to most tenth graders, but which we believe it is important for tenth grade teachers to know. The notes probe deeply into the beginnings of the subject, but do not cover much of the material of the standard tenth grade geometry course. In particular, the notes do not cover such topics as parallels, circles, areas, Pythagorean theorem, analytic geometry, etc., whereas all of these will be treated in the SMSG text book.

It is assumed that teachers who study these notes have good backgrounds in axiomatic geometry. In particular some familiarity with Euclid's Elements is presupposed, and the teachers should have access to these. (Heath's translation in Everyman's Library (E.P. Dutton) is convenient.) The notes contain only occasional references to the common high school treatments of geometry, but the readers should continually make comparisons of the two types of treatment, especially in the proofs of the more familiar theorems.

Although we have tried to anticipate the content, order, notation, etc. of the proposed text book there is bound to be some divergence. However, we are confident that we are presenting the spirit of the new course, and that anyone who understands the material in these notes will be able to use the text book in an intelligent and interesting manner.

The instructor in a summer course for teachers is urged to spend most of the time on Chapters 4 - 7. These contain the material on the Ruler and Protractor Axioms, and the theory of separation or order of points on a line, lines in a plane, and planes in space. Much of this material is unfamiliar to many teachers.

Chapters 1 - 3 are introductory in nature, and can be covered swiftly at the beginning, and referred to from time to time as the course progresses.

These notes are by no means a polished work, and it is expected that the instructor will use good judgement in deciding which parts to amplify, which to slight, what extra material to put in, etc. Some exercises have been included but many additional ones will have to be supplied. The starred theorems (e.g. Theorem $3.1^*$) can be used as exercises, and it is intended that as many as possible of these be proved. Most of the early ones are used in later proofs, and their ommission would leave serious gaps in the presentation.

None of our proofs are accompanied by diagrams. This was done deliberately, partly to urge the reader to draw his own diagram and partly to emphasize that a logical proof should be independent of any diagram. The ommission does not mean that we wish to minimize the importance of drawing a figure to fix the ideas in the mind. The readers should make constant use of diagrams as an aid to understanding the theorems and discovering proofs.

Future work by the SMSG on the text book, the teachers' manual, and the teacher training manual can profit greatly from comments and criticisms of these notes. You are urged to send your suggestions to

School Mathematics Study Group
Drawer 2502A Yale Station
New Haven, Connecticut

## Preface to Second Revision

Several changes have been made in this edition to improve the material and bring it more into line with the text book.

1. The notations $\overleftrightarrow{AB}$ for a line and $AB$ for a distance have been adopted. On the other hand it did not seem worthwhile to change $m(\angle ABC)$ to $m\angle ABC$ and arc $AB$ to $\widehat{AB}$.

2. The distance and separation postulates have been reworded to conform to the text book. However, to change the order of presentation of these postulates to that used in the text book would have necessitated extensive alterations in Chapters 3 and 4. The text book order was dictated by pedagogical considerations, principally the desire to avoid indirect proofs in the first theorems, which do not apply to this book. It was therefore decided to leave the order as in the earlier editions. The postulates thus differ in numbering as follows:

| | These Notes | Textbook |
|---|---|---|
| Incidence postulates | 1, 2, 3, 4, 5 | 1, 5, 6, 7, 8 |
| Distance postulates | 6, 7, 8 | 2, 3, 4. |

3. Chapters on Parallels and on Area have been inserted to clarify the position of these topics in our presentation.

4. Numerous minor changes, insertions, corrections, etc. have been made, and one significant error has been corrected (Section 4 of Chapter 5).

The user of this book will find much additional material in the text book, including simple exercises and expository material in the text book proper and supplementary reading in the Teachers' Commentary, the Appendices, and particularly the Talks to Teachers.

In this book we present Euclidean geometry as a mathematical theory. We have left aside all applications of geometry to questions in physics and to other branches of mathematics. This is not to say that we regard these applications as unimportant for the teacher to know and to use in the classroom; in fact, it would be a mistake to teach geometry to high school students without bringing in some of the significant applications, especially to elementary physics. The reader of this book will find abundant supplementary material on applications of elementary geometry in the following books:

G. Polya, Mathematics and Plausible Reasoning, Princeton University Press, vol. 1, especially the chapter on "Physical Mathematics".

R. Courant and H. Robbins, What is Mathematics?, Oxford University Press.

H. Rademacher and O. Toeplitz, The Enjoyment of Mathematics, Princeton University Press.

# CONTENTS

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Chapter Page

1. Historical Introduction . . . . . . . . . . . . . . . 1

2. Logic . . . . . . . . . . . . . . . . . . . . . . . . 37

3. Points, Lines, and Planes . . . . . . . . . . . . . . 63

4. Real Numbers and the Ruler Axiom . . . . . . . . . . 69

5. Separation in Planes and in Space . . . . . . . . . 107

6. Angles and the Protractor Postulates . . . . . . . . 115

7. Congruence . . . . . . . . . . . . . . . . . . . . . 131

8. Parallelism . . . . . . . . . . . . . . . . . . . . 145

9. Area . . . . . . . . . . . . . . . . . . . . . . . . 153

10. Circles and Spheres . . . . . . . . . . . . . . . . 169

## Chapter 1

## Historical Introduction

1. The Glory that is Euclid's. "Euclid is the only man to whom there ever came or ever can come the glory of having successfully incorporated in his own writings all the essential parts of the accumulated (mathematical) knowledge of his time."[1]

He was the most successful text-book writer that the world has ever known. Over a thousand editions or revisions of his geometry have appeared since the advent of printing, and his work has dominated the teaching of the subject ever since his Elements appeared, first in manuscript form and then in the form of revised text-books. These revisions always kept the essential ideas as developed by Euclid. Euclid's Elements was, for the most part, a highly successful compilation and systematic arrangement of the works of earlier writers. Euclid accumulated the mathematical knowledge which had developed over a period of some 300 years, during which deductive reasoning in mathematics had evolved, and organized this material into the oldest scientific text-book still in actual use.

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[1] D. E. Smith, History of Mathematics, Ginn, vol. 1, p. 102.

The material of the first four Books can be traced back to the school of Pythagoras (572-501 B.C.); Books V and VI treat the theory of proportion including the method of exhaustion, developed by Eudoxus (c. 370 B.C.), and Book X presents the theory of irrationals as developed by Theaetetus (c. 375 B.C.). There is little doubt, however, that Euclid had to supply a number of proofs and to complete or perfect others. The chief merit of his work lies in the skillful selection of the propositions and their arrangement in a logical sequence presumably following from a small number of explicitly stated assumptions. Euclid referred to these assumptions as definitions, common notions and postulates. Indeed, many of our modern texts[1] fail in some respects in which Euclid succeeded.

Perhaps some of the early phrases of this chapter are a slight exaggeration, for some evidence is missing. No original copy of Euclid's work exists, and it is difficult to tell what is due to Euclid and what is due to the early revisionists. Recent archeological studies indicate that through the various early hand-written editions, many changes were made in the "fundamentals" - definitions, common notions, and postulates - so that we do not know precisely what Euclid wrote. These studies do indicate that there was early dissatisfaction with some of the preliminary ideas; this dissatisfaction persists even today. The propositions usually appear as Euclid wrote them, since all such early manuscripts are in general agreement on this score. But there is no doubt that Euclid's work is the earliest attempt at a systematic arrangement of definitions, common notions (now often called axioms), postulates and propositions which presents mathematics as a logical deductive science. The subsequent influence on all scientific (not merely mathematical) thinking can hardly be overstated.

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[1] Current American texts on plane geometry are based on Books I, III, IV, V, and VI of Euclid's Elements; texts on solid geometry are based on Books XI and XII. The best and most reliable reference work now available is the monumental three volume work, recently reprinted, by T. L. Heath, The Thirteen Books of Euclid's Elements, Dover Publications, 1956.

In order to understand and appreciate fully the contributions of Euclid to mathematical thought, we must go back into the history or even pre-history of mathematics. Geometry, as an intuitive or factual body of knowledge, grew out of natural necessity. Indeed, the very word "geometry" means "measure of the earth". Many geometric facts were collected by early civilizations in Egypt, Babylonia, China, and India. The facts were stated without any indication of any process of deductive reasoning. Some of the ideas were precise, others were approximate and were arrived at from experience, and some were just guesses; the results of experience or pure guesses were not always correct. The Theorem of Pythagoras, for example, at least in one form, was known to the Babylonians at least as early as 1600 B.C. Tablets, dated that far back, giving tables of values of integral solutions of $a^2 + b^2 = c^2$, have been uncovered. Being convinced, by experiment, that "The square on the hypotenuse of a right triangle is equal to the sum of the squares on the legs" is one thing, and proving this logically from explicitly (or even implicitly) stated assumptions is another thing. We suggest an analogy from arithmetic: it is one thing to know that the product of two odd numbers is odd, but it is a "horse of a different color" to give explicit definitions and postulates and then prove this fact.

The origin of early Greek mathematics is clouded by the greatness of Euclid's Elements, because this work superseded all preceding Greek writings on mathematics. After the appearance of the Elements, all earlier works were thenceforth discarded. One of the later commentators, Proclus (c. 460 A.D.), who did so much towards preserving Euclid's work for us, in contrasting Euclid with earlier writers - no doubt believing in the 'infallibility of Euclid' - stated in effect:

"The selection and arrangement of the fundamentals was complete, clear, concise, and rid of everything superfluous. The theorems were presented in general terms, rather than as a number of special cases, and in all ways, Euclid's system was superior to all the rest."

We do know some facts about early Greek mathematics. The history of demonstrative geometry properly begins with the Greek geometer, Thales of Miletus (640-546 B.C.). His actual contributions to geometrical knowledge were few, but he first recognized the necessity of giving a demonstration based upon a logical sequence of ideas. He took the first step in raising geometry from a set of isolated facts of observation and crude rules of calculation, concerning material things, to a logical consideration of geometric concepts abstracted from these material things. He was followed by Pythagoras and his School, whose main contribution was that mathematics was studied from the intellectual viewpoint. The School of Pythagoras employed the deductive process of reasoning exclusively and systematically, and thus raised mathematics to the rank of a science, despite the fact that their whole philosophy was shrouded in the mysticism of whole numbers. They distinguished mathematical theory from practice (which they disdained, and proved fundamental theorems of plane and solid geometry, as well as theorems of the theory of numbers. To their dismay, they also discovered and proved the irrationality of $\sqrt{2}$. This very discovery, being directly contrary to Pythagoras' preconceived mystical and indefensible concept of the relation of numbers to the universe, was the cause of the downfall of his own school of thought.

We shall mention only two of the many geometers that followed this period. The renowned paradoxes of Zeno (c. 450 B.C.) initiated another crisis in mathematical thought. They are concerned with the difficult problems of continuity and the riddles of the infinite. Even today the problems of continuity are real stumbling blocks for school-boys (or school-men), and the riddles of the infinite have not yet been completely unraveled by mathematicians and logicians. As far as high school geometry is concerned, the problems of continuity were resolved by Eudoxus (408-355 B.C.) in his method of exhaustion. In modern terminology, we would refer to this method as the theory of inequalities, which in turn is intimately connected with the theory of limits. In our American texts such topics are often completely overlooked or relegated to the appendix (that part which is usually left out). In our better texts after some rationalization, proofs involving such topics are replaced by appropriate postulates.

Then came Euclid. His Elements (c. 300 B.C.) unified the work of many scholars and systematized the known[1] mathematics of the day. The definitions and assumptions, the arrangement, the form, and no doubt the completion of many partially developed topics are (as far as we know) due to Euclid, although he leaned heavily on the shoulders of Pythagoras and Eudoxus. Euclid set himself the task of finding an adequate and universally acceptable set of postulates for geometry, and at the same time, of avoiding a proliferation of assumptions. Holding down the number of postulates adds to the "fun" when some conclusion is reached with less price in the number of assumptions; it may also increase the number of interpretations or models of the postulate system, or make it easier to check that a model actually satisfies the postulates. (See Section 6 of Chapter 2 on Postulates or the article on Finite Geometries in the Commentaries for Teachers.) Our modern texts do often contain a proliferation of assumptions, many of which are repetitious in the sense of non-independence. Euclid tried to avoid this. In our modern texts, the criterion, maybe advisedly, seems to be: 'If it is difficult to prove, assume it.' Too often the assumption is never explicitly stated. Euclid tried to avoid this also. However, some compromise must be made between having just a few postulates and presenting a large number of postulates, so that the theorems proved are those that are most readily understood by the audience for which the material is intended. Euclid did not write for school-boys; but for the scholars and philosophers of his day. His work showed a seriousness of purpose and a desire to be rigorous and to avoid the use intuitive geometry. He even demonstrates the correctness of his constructions before using them, and he is not afraid to treat incommensurable magnitudes in a logical fashion. He was interested in the systemization of geometric facts, not in their discovery. His work showed no interest in the analysis of a proof, but rather in its synthesis in a rigid form: proposition, hypothesis, proof, conclusion. He tried to push aside the geometric facts gained by experience and their practical applications, and placed emphasis upon logical deductions. "Though experience is no doubt a good teacher, in many situations it would be a most inefficient way of obtaining knowledge ··· . The method of trial and error may be direct, but it may also be disastrous."[1] To this quotation we may add, not only that it may be disastrous, but it has been. The history of mathematics is replete with incorrect statements based on limited experience or experiment. This does not mean that we should overlook the role of experiment, either physical or mathematical, in suggesting facts. We must merely make sure that any suggested fact is given a logical proof.[2]

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[1] This work does not consider the subject of conics. There is evidence that some knowledge of this subject was available, but there is also some evidence that this topic was included in some of the "lost Books" of Euclid.

Euclid recognized the importance and necessity of starting with appropriate definitions and assumptions. He went to unnecessary and inadvisable lengths to define every term, although he was acquainted with Aristotle's statement: "It is not everything that can be proved. You must begin somewhere." Euclid recognized this with regard to his axioms and postulates but overlooked the corresponding idea with regard to his definitions.

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[1] Morris Kline, Mathematics in Western Culture, Oxford University Press, p. 24.

[2] For excellent discussions of the role of experiment and other types of plausible reasoning in mathematics see the books by G. Polya, How to Solve It, and Mathematics and Plausible Reasoning.

Unfortunately, most modern text-book writers haven't recognized it yet. It is not our purpose to discuss his definitions here - that will come later - but to point out that they are not satisfactory from either the modern point of view or Euclid's ideal and purpose. They cannot be used in the devolpment of the superstructure (the Propositions) upon a strictly logical basis. Euclid's common notions, often called axioms, are essentially general statements which correspond to the usual axioms of equality and addition of ordinary arithmetic. Euclid did not refer to them as "self-evident truths"; this connotation was due to later writers, who were not as expert as Euclid. If, however, we accept these definitions and common notions upon an intuitive and descriptive basis as abstracted from our physical universe of common experience, they may guide us to later precise definitions and assumptions, which can be used as a basis for a strictly logical development of geometry. This is the basis of Hilbert's[1] axiom system, by means of which Euclid's Elements can be made logically correct. It should be understood that there are no "false" Propositions in Euclid's Elements, only incomplete proofs and statements that do not follow logically from his stated premises, but which depend upon his preconceived ideas based upon intuition and experience. A number of these omissions will be discussed later in Section 2.

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[1] David Hilbert, Grundlagen der Geometric, 7th and final edition, Teubner, 1930. Several editions of the English translation (Foundations of Geometry, tr. by E.J. Townsend) have also appeared. For a condensed version for plane geometry, see Eves and Newsom, An Introduction to the Foundations and Fundamental Concepts of Mathematics, Rinehart, 1958, pp. 87-88. See also Section 2 of Chapter 3, where the incidence postulates stated are essentially those given by Hilbert. Other references and details of Hilbert's postulates will be given later.

The assumptions that were fundamentally of a geometric character Euclid called Postulates,[1] and it was here that Euclid showed his mathematical acumen. There were five such Postulates.

I. A straight line can be drawn from any point to any point.

II. A finite straight line can be produced continuously in a straight line.

III. A circle may be described with any center and passing through a given point.

IV. All right angles are equal to one another.

V. Will be considered later.

Although he didn't say so, there is no doubt that Euclid implied the uniqueness as well as the existence of the corresponding line and circle. Euclid did not pay any attention to the tools, ruler and compass, theoretical or practical, by means of which these constructions could be made. As has been already pointed out, he was interested only in logical deduction and not the applications of geometry. Of course, we cannot actually draw a line or a circle in a physical sense, but we can draw very good models of them. The entities of geometry are mental constructs and the drawings are physical objects with roughly similar properties or substitutes for them. The heuristic value of these models was not overlooked by Euclid and should never be under-rated by us. A well-drawn diagram has enormous heuristic value; it is essential in the "discovery" process and in the analysis, and it is even very useful in the synthesis or formal proof à la Euclid, but it cannot be used to provide a logical proof of the statement. Indeed, the fact that the proof must use the model is prima facie evidence that something is wrong or missing. The best two illustrations are Euclid's proof of the "Theorem of the Exterior Angle" and of the Proposition: "If two planes have a point in common, they have a line in common."

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[1] Today, mathematicians make no distinction between the terms: axiom, postulate, assumption, agreement, and principle, as long as they are merely statements which are assumed without proof.

In his second postulate Euclid recognized the infinite character of the line, and although he did not state so precisely, he used the postulate in the sense which required that the length of the line be infinite. All the definitions, axioms, and postulates so far were relatively simple of comprehension and fully in accord with experience, and no one ever questioned them for over 2000 years. But the fifth postulate - Euclid's Parallel Postulate - was of a different character.

V. "If a straight line falling on two straight lines makes the interior angles on the same side together less than two right angles, then the two straight lines, if produced indefinitely, meet on that side on which the angles are together less than two right angles."

This postulate was not accepted without misgivings, and much of the history of Euclidean geometry after the fall of the School of Alexandria until today is concerned with the attempts and failures to prove this postulate on the basis of the other assumptions. We shall discuss some of these attempts in the next section.

The introduction by Euclid of his Parallel Postulate was no accident. It was a monument to Euclid's insight and skill as a mathematician and logician. The evidence is in the Elements. He tried to prove everything he could without it; he even introduced peculiar Propositions with but one intent - to prove all he could about parallel lines without the Parallel Postulate. This postulate is essentially the converse of Proposition 17, Book I of Euclid's Elements, which in brief form is:

"If two lines cut by a transversal do meet, then the sum of the angles is less than two right angles."

It is true that the proof of this theorem depends upon Proposition 16, the "Theorem of the Exterior Angle", whose proof is not complete and cannot be completed on the basis of Euclid's explicitly stated assumptions. But even the best mathematicians did not realize this for some 2000 years. They were so intent upon "proving" the Parallel Postulate that they over-looked the other errors and the possibility that it could not be "proved". It would be much simpler to develop geometry if Postulate V were used immediately after Proposition 17, but because Euclid didn't, modern text-books do not either. Euclid went on as long as he could without it; modern foundation theory often does the same thing. In Proposition 28, Euclid proved the contra-positive - the opposite of the converse of Proposition 17, and then used his parallel postulate to prove the opposite of Proposition 17, which is also the converse of Proposition 28. It is indeed a peculiar order, but motivated by Euclid's desire to prove everything he could and postpone the use of Postulate V as long as he could. In between Propositions 17 and 27, Euclid developed the theory of inequalities with respect to one or more triangles. Euclid was willing to pay the price of harder work to obtain Propositions 18 to 28 without recourse to Postulate V -- and for the moment the contents of these Propositions is irrelevant -- the numbers are relevant because they tell how long Euclid postponed the use of this Postulate. If nothing more is learned from this analysis, it should point out the extreme care used by Euclid in order to systematize and organize geometry into a complete and logically correct structure.

In summary, the outstanding contribution of Euclid's Elements lies in the development of the modern mathematical method - the hypothetical-deductive method of modern mathematics. We owe much to Euclid because he possessed the ideal of placing mathematics on an unimpeachable logical basis. He demonstrated how much knowledge can be derived by reasoning alone - for he gave hundreds of proofs based upon a relatively few assumptions, and it was through his *Elements* that later civilizations learned the power of reason.

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2. *The Mistakes of Euclid*. Over a period of almost 2000 years, many mathematicians accepted Euclid's Parallel Postulate with misgivings, and attempted to prove this postulate on the basis of his other axioms and postulates. Many "false proofs" were published, or unwittingly, "proofs" were given which depended upon *assumptions* which are logically equivalent to Euclid's postulate. These attempts and failures finally led to the discovery of what we now call non-Euclidean geometries, which plainly showed the importance and necessity of Euclid's Parallel Postulate (or some logically equivalent one) for the completion of Euclid's work. But they did more than that. They opened up wide vistas of mathematical progress uninhibited by the doctrine of the infallibility of Euclid.

The first real progress was made in 1733 by a Jesuit Priest and Professor of Mathematics at the University of Pavia, Gerolano Saccheri, who, however, reputiated his own achievements, and entitled his work: "Euclides ab omni naevo vindicatus," or freely translated: "Euclid is free of every blemish." Saccheri denied an assumption that is logically equivalent to Euclid's Parallel Postulate and kept all the rest. He developed a logically consistent body of theorems for a geometry which differs from that of Euclid. But he was so convinced of the infallibility of Euclid, that, in a final chapter, he lost himself in the morass of philosophical meanderings in the realm of the infinitesimal and rejected all his own correct reasoning and concluded with: "Euclides ab omni naevo vindicatus" - when indeed the Fifth Postulate was not even one of the mistakes of Euclid. If Saccheri had had a little more imagination and had not been so convinced that there could be no mistakes in Euclid, he would have anticipated by a century the discoveries of Gauss, Lobachevski, and Bolyai. Indeed the Parallel Postulate was not one of the mistakes of Euclid, but it was one of his crowning mathematical achievements. It is not our intent to go deeply into the subject, but to make a few pertinent comments. Gauss had developed many ideas along this line by 1800, but he had published nothing on the subject. But Lobachevski in 1823, and in later writings, had the courage of his convictions. Although the imprint of Saccheri's work is plainly visible in his writings, his attitude was different, and he gave a complete development of Hyperbolic Geometry. About the same time, the work of the younger Bolyai was sent by his father to Gauss. Gauss replied that he had been in possession of much of this material a long time, only to be accused of plagiarism. But all of these writers had still placed too much faith in Euclid, and it wasn't until Riemann, in 1854, published his famous dissertation that the true situation became apparent. Riemann started from an entirely different point of view, that of differential geometry, and showed the existence of so-called Elliptic Geometry, which his predecessors had rejected, and pointed out that other postulates of Euclid needed careful scrutiny. The importance of the work of these men does not lie entirely in the discovery of new geometries, but also in the fact that it caused a crisis in mathematical thinking which led to a critical examination of Euclid's Elements and to the discovery of many mistakes of Euclid and to methods of correcting them.

Felix Klein[1] wrote in 1908:

"The ideal purpose which Euclid had in mind was obviously the logical derivation of geometric theorems from a set of premises completely laid down in advance. But Euclid did not, by any means, reach his high goal. Nevertheless, tradition is so strong that Euclid's presentation is widely thought of today as the unexcelled pattern for the foundations of geometry."

Let us now examine some of the mistakes of Euclid, not for the purpose of criticism, but so that we may avoid them in our presentation of geometry.

(1) Euclid tried to define every term.

Let us look at one illustration. "A point is that which has no part." Surely this does not tell you what a point is. Other mathematicians of his period did not do much better. Pythagoras said:

"A point is a monad having position."

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[1] Elementary Mathematics from an Advanced Standpoint. See vol. 2, pp. 188-208. See also A E. Meder, Jr., What is Wrong With Euclid, Mathematics Teacher, Dec., 1958.

And what do you mean by position? Let us make up a definition: Position is a property possessed by a point! This should be enough to convince you that we had better let the term 'point' remain undefined. It is only necessary to examine Euclid's definitions of point, line, and plane to discover that they play no role whatever in the logical development of geometry. We must start somewhere with undefined terms if we wish to avoid circumlocutions, and here is a good place to begin.

(2) Euclid's definitions are not always precise and meaningful.

"A line is length without breadth."

If we know what is meant by length and breadth independent of any connection with a line (curve), this might be precise, but as it stands it is nonsense. It tries to identify a geometric entity, a line, or perhaps a collection of entities, a set of points, with some attributes of measurement, which themselves are meaningless without knowing what a line is.

In the above definition, the word 'line' is used in the sense of 'curve', including 'straight line', and the latter term is then defined as follows:

"A straight line is a line which lies evenly with respect to its points."

This statement is wholly obscure. If you try to explain it in terms of motion, you only complicate matters by bringing in more undefined terms which are extraneous to the subject matter. That is, you might prefer some modern (sic) definition like this: "A line (curved line) is the path of a moving point." First, the idea of physical motion has been introduced, and second, what do you mean by path? If you try to side-step the difficulty by saying we merely mean change of position, you are back where Pythagoras began.

Perhaps you have heard this often: "A straight line is the shortest distance between two points." As a definition it is entirely unacceptable. The very concepts of line or line segment and 'distance' are radically different. It is true that Euclid proved the Triangle Inequality Theorem, but that comes many pages and some twenty propositions after the definitions were given. And what do you mean by the length of a curved line? If you know anything about limit theory or the integral calculus, you might supply an answer, but it is too late. Perhaps you are convinced, anyway we are, that the term 'line' had better remain undefined.

These and many other ill-stated definitions may be harmless if properly fenced off in the department of pictorial representation or informal geometry. The postulates list those assertions from which all conclusions in this branch of mathematics will follow. It is merely in the logical development of geometry that such ill-stated definitions are worse than useless. This is discussed further in Chapter 2 under the headings of Descriptive Definitions, Postulates, and Explicit Definitions.

(3) Euclid's postulates are not always stated precisely.

"Postulate 1. It is possible to draw a straight line from any point to any point."

'Draw' has a physical connotation but we will credit Euclid as meaning it in the sense of 'there exists'. Such ideas were current in the philosophy of his day. But as Euclid used the postulate, he meant more than that. He meant there is one and only one such line. It is not difficult in this case to say what you mean, and that is precisely what we intend to do.

"Postulate 2. It is possible to produce (extend) a finite straight line continuously in a straight line."

This postulate was misunderstood because Euclid did not indicate what he meant by 'continuously'. To understand the difficulty let us talk in terms of measurement. If we start with a segment (Euclid's finite straight line) of unit length and produce it $\frac{1}{2}$ a unit, and then $\frac{1}{4}$ a unit, and then $\frac{1}{8}$ a unit, and so on ad infinitum, each time extending it by one-half the previous extension, we will be producing the line continuously, but that is not what Euclid meant nor how he used it. But he was familiar enough with Zeno's paradoxes that he should have avoided his loose statement. No doubt Euclid meant that the segment could be extended by any amount (indefinitely), but so could a circular arc. But there is a difference. In the case of the line, Euclid meant that we would never return to any point we had before, while in the circle we would. This distinction was not clearly recognized until the time of Riemann's famous dissertation of 1854, so Euclid is excusable, but not his modern imitators. Euclid's difficulty cannot be resolved without some consideration of the concept of 'between', which Euclid entirely overlooked, except on a purely intuitive basis. This difficulty was recognized by Eudoxus before

Euclid, and by Archimedes after Euclid, and we must give proper attention to this concept of 'between' in our presentation of geometry.

There are similar difficulties in other postulates and common notions, but we need not point them out here. We merely need to heed the warning that if we expect to present geometry on a strictly logical basis that it is necessary to say what you mean, or some one will misinterpret what you say.

(4) The postulate system is incomplete.

We have already given some indication of this with regard to the Postulate of Extension and the lack of consideration of any Postulates of order. But the same indictment can be made because the proofs of some propositions are logically incomplete due to the fact that the assumptions upon which they are based were not included in the Postulate system. Let us look at some of the proofs in detail.

Proposition 1. Book I.

"On a given finite straight line to construct an equilateral triangle."

The essence of the construction is to draw a circle with center A which passes through B, and one with center B which passes through A.

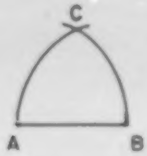

Fig. 1

*If* the circles meet at C, then the triangle ABC is equilateral. Notice our emphasis on the word 'If'. Since we must take the terms line and plane as undefined, we really do not know precisely what they mean, although we may have pretty good models for them as in Fig. 1. Perhaps you construct such a model, using a line segment of relatively small length, and say: "I can see that they intersect. If you can't see that, you are stupid, or mentally deficient, or just stubborn." If we keep on producing the line segment, it won't be long until we can no longer construct the physical model, and we can not see if the circles intersect. The Proposition is stated for any finite line segment, and the proof must be for any segment.

Suppose we try the same construction on the surface of a sphere, where we have a different model, the sphere being called a "plane", and a great circle being called a "line", where all we need to guarantee is that these words in quotes satisfy the postulates that are used. Upon examination this will be found valid if the "line" segment has a length less than one-half that of a great circle, that is, if we restrict our model to a hemisphere. First we use a "line" segment of relatively small length, and we see that the circles intersect. We might conclude that the Proposition (it is really an existence theorem stated in different language) is also valid on the sphere. If the sphere had a radius of a mile, or say 4000 miles, we probably

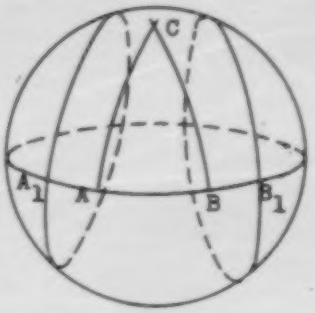

Fig. 2

could not preceive the difference between it and a plane surface. On the small sphere, let us extend the "line" segment $\overline{AB}$ a little longer and little longer (Postulate 2) but always keeping A and B on the same hemisphere, and try it again, as with $\overline{A_1B_1}$ in Fig. 2. If $\overline{A_1B_1}$ exceeds a certain length the circles do not intersect. Is this a paradox? No, just an error of omission in the original proof. Of course, we must assume the circles are continuous[1] curves, curves without gaps, but that is not enough. The circles always intersect in the Euclidean plane properly considered but do not always intersect on the sphere, and this difference must be established by proof or assumption. What is needed here is a circle axiom in the plane which will guarantee the intersection of circles under appropriate restrictions. This circle axiom is not only a continuity axiom, but is also equivalent to the converse of the Triangle Inequality Theorem, which appears as Proposition 20, Book I of Euclid's Elements in the form:

"In any triangle two sides taken together in any manner are greater than the remaining one."

In Proposition 22, Euclid proposed a construction problem:

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[1] The concept of continuity is a difficult one, not only with respect to points on a line, but also with respect to our number system. This concept is often hidden in our Postulate System, but the teacher needs to be aware of some of its implications.

"Out of three straight lines, which are equal to three given straight lines, to construct a triangle: thus, it is necessary[1] that two of the straight lines taken together in any manner should be greater than the remaining one." "... it is necessary ... ," but nothing is said about it is sufficient, and "aye, there's the rub." This sufficient condition is one form of a circle axiom. Using the language of Euclid, but not quoting, we could take as an axiom:

Circle Axiom. Out of three straight lines, which are equal to three given straight lines, such that two of them taken together in any manner are greater than the remaining one, it is possible to construct a triangle.

This circle axiom can be stated in many different and better ways for the plane, but it is not valid for the sphere. It is sufficient to note that the Triangle Inequality Theorem (Proposition 20) is true but incomplete for a spherical triangle. It is really a very special case of the Circle Axiom that is needed to complete Proposition 1, because for the stated construction we would use $c + c > c$. But our second model suggests that the converse is false for the sphere. To prove a statement is false, one counter-example is enough; to prove it is true, a dozen special cases are not enough. We assume the circle axiom for the plane based on our limited experience and this, like all other postulates about a plane, puts definite restrictions on what the undefined term plane means.

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[1] The emphasis on necessary is supplied here.

Let us now turn our attention to another concept with respect to which the postulate system of Euclid (and his modern imitators) is incomplete, that of motion[1]. Some of the proofs in the theory of congruent triangles depend upon the physical concept of motion which is entirely extraneous to Euclid's geometric development. It is true that we *can define* what Euclid meant by motion, or that we can take the concept as *undefined*, *subject to a set of assumptions*, but Euclid did neither and many of our modern texts do not do it properly either. These texts do essentially make an existence axiom: "There exist motions that do not change size or shape," but that is not enough. Euclid, himself, recognized part of the difficulty and avoided the use of motion and its concomitant, superposition, whenever he could give a proof by other methods. He used it sparingly and often proved propositions without it, even though its use would provide a simpler "proof". It is true that a mathematical system of postulates and definitions for motion can be made, but this point of view is difficult to develop, and uses explicitly the theory of congruent triangles, and, hence, could not be used to develop the theory of congruent triangles. It is no wonder that Birkhoff and Beatley, in the Preface to their *Basic Geometry* call such "proofs" *demoralizing*, and this point of view is upheld by any realistic mathematician. But the situation in

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[1] For a further discussion of this concept, see the Appendices to the text, and Talks to Teachers in the Commentaries.

Euclid's Elements involves another difficulty. Klein says: "The only conceivable purpose for Euclid's Propositions 1, 2, and 3 is to avoid the use of physical motion in order to prove Proposition 4, the Side-Angle-Side Theorem. But then Euclid did not use them, perhaps recognizing that they were not sufficient for that purpose."

If you wish to apply the theory of congruent triangles to triangles in different planes, there is still another error of omission in Euclid's Elements, even if you grant the use of motion. It depends upon Euclid's proof in one of the later books of the Proposition:

"If two planes have a point in common, they have a line in common."

The proof of this Proposition is incorrect. It depends upon the following cited statement:

"If two planes have a point in common, they have a second point in common."

Why? No reason is given, and no reason based upon any of his previous work can be given. Of course, we never saw two planes that did not have this property, really we never saw two planes, although we have seen pretty good models of them. However, let us consider another model. Two spheres (recall that the (S.A.S) Theorem is also valid on the sphere) are placed so they have and retain one and only one point in common, and let us imagine these spheres to grow in size until each has a radius of 4000 miles, say. Do you think you could perceive the difference between the spheres and two planes anywhere near the common point? Of course not, but these spheres have only one point in common. Euclid's assertion is a pure assumption about the properties of planes in 3-space. Actually it is false concerning planes in 4-space. Of course, Euclid had no such conception available. Since this Proposition cannot be proved on the basis of the other Postulates, it must be taken as a Postulate, and this is just what we do.

The most serious mistake in Euclid's Elements is the complete omission of any consideration of order of points on a line, that is, the concept of betweenness, and of separation of a plane by a line. This accounts for the errors of omission in the 'proof' of the Theorem of the Exterior Angle. This theorem is fundamental for Euclid's development of the theory of inequalities related to a triangle, and to his theory of parallel lines. The crucial point of the proof depends upon the proof that a certain constructed point lies in the interior of an angle. But since no proper consideration of order relations or separation axioms is included in the Elements (or by most of his imitators), no proper proof can be given until appropriate Postulates and Definitions are given. These are given in our program for the development of geometry.

It is no wonder, then, that Klein states:

"So many essential difficulties present themselves, precisely in the first theorems of the first book of the Elements, that there can be no talk about the attainment by Euclid of his ideal."

Nor is it any wonder that Hilbert, Birkhoff, and others, as individuals, and the Commission on Mathematics, The Illinois Study Group, and the School Mathematics Study Group reached the conclusion that to attain the ideal goal of Euclid, we must have a fresh start. It will not be Euclid, but it will be Geometry that is completely vindicated.

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3. The Program for Geometry. After the mistakes of Euclid have been pointed out, it is not too difficult to correct them by introducing the needed assumptions. The best known procedures are based upon the works of David Hilbert[1] and G.D. Birkhoff[2]. Hilbert's program was to stay as near the form of Euclid as possible and to supply precise postulates which can be made the basis of correct proofs of all the Propositions of Euclid's Elements[3]. The entities point, line and plane, and the relations incidence, between and congruent are taken as undefined but limited by precisely stated postulates. The first postulates are concerned with the incidence

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[1] David Hilbert, Grundlagen der Geometric (Foundations of Geometry). A number of both German and English editions are available. See also Eves and Newsom, An Introduction to the Foundations and Fundamental Concepts of Mathematics, p. 87.

[2] G.D. Birkhoff, A Set of Postulates for Plane Geometry, Based on Scale and Protractor, Annals of Mathematics, vol. 33 (1932), pp. 329-345. See also Birkhoff and Beatly, Basic Geometry, 1940.

[3] A text for high school students based upon this program has recently been published: Brumfiel, Eicholz and Shanks, Geometry, 1960, Addison-Wesley.

of points and lines, points and planes, lines and planes, and two planes, and fill the gaps left to the imagination by Euclid[1].

The next set are the axiom of order, involving the concept of between, in order to give the points on a line the same characteristics that we usually relate to real numbers. These ideas are then related to points in a plane in order to develop the notion of separation of a plane by a line, a notion often used by Euclid, but about which Euclid said nothing in his "Elements". Hilbert used an axiom due to Pasch which can be stated as follows:

Axiom of Pasch. "A line which passes through a point between two vertices A and B of a triangle ABC, either passes through a vertex, or a point between A and C, or a point between B and C."

The next set contains the axioms of congruence concerning congruent segments and congruent angles, stated in a manner to suggest their analogy with the relation of equality and the operation of addition that we usually associate with real numbers. These three sets are necessary to fill in the gaps left by Euclid. To complete the postulate system, a parallel postulate and a continuity postulate are required. Hilbert used the Playfair form of the parallel postulate[1] and an axiom of continuity known as the Law of Archimedes[2].

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[1] It is not expected that the reader be acquainted with the works of Hilbert and Birkhoff. However, Hilbert's incidence postulates as used by SMSG are stated at the beginning of Chapter 3. The purpose here is merely to start a discussion of the points of view of these two leaders, and to state how SMSG drew upon both of these presentations.

Euclid did not write for school-boys but for philosophers and scholars of his day. Neither did Hilbert write for school-boys. In part, the Hilbert (or synthetic) approach is too sophisticated for a beginning course in geometry. This is why we have been forced to the conclusion that a tenth-grade course based upon Hilbert's Foundations of Geometry would, in our opinion, be so unteachable as to be ridiculous. This does not mean that we reject his ideas entirely. Indeed we use many of Hilbert's ideas. We accept the fundamental idea that point, line, and plane should remain undefined and adopt Hilbert's Incidence Axioms, essentially as he gave them.

We adopt, however, the point of view of Birkhoff, that we should arithmetize geometry as much as possible, and build upon the student's knowledge of arithmetic, elementary algebra and his ability to use a scale and protractor. This requires a careful

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[1] For a statement of this postulate see p. 32.

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[2] For a geometric statement of this postulate, see Eves and Newsom, loc. cit., p. 88. An arithmetic equivalent is as follows: If $a$ and $b$ are positive numbers, there exists a positive integer $n$ such that $na > b$.

statement and discussion of the Axioms of Linear Measures and Angular Measure which are not beyond the understanding of a tenth-grade student. It is assumed that the student is already familiar with those properties of real numbers that are needed, or these properties are stated (as axioms) on an informal basis. This arithmetic approach not only has the advantage that it builds upon what the student already knows, but it lays the foundation for the early introduction of analytic geometry. When it is simpler to use definitions or postulates given by Hilbert than those given by others, we have no hesitation in using them. But many of Hilbert's ideas appear as theorems that can easily be proved by the Birkhoff approach. In particular, the notions of between and segment are closely associated with corresponding ideas of arithmetic. It is very difficult to prove the Axiom of Linear Measure from the Hilbert or similar approaches. If you need to be convinced of this fact you need only examine how it is done in H.S.M. Coxeter's, The Real Projective Plane, Chapter 10, or to study H.G. Forder's book, The Foundations of Euclidean Geometry. This points out that the Axioms of Linear Measure and Angular Measure are very powerful and desirable tools for the development of geometry.

We have adopted a Separation Postulate[1] (of the plane by a line) instead of the Axiom of Pasch because it is more directly applicable to the theorems we wish to prove, and it is more nearly related to the ideas of inequality useful in analytic geometry. It is this Postulate, for example, which permits us to supply the missing steps in Euclid's proof of the Theorem of the Exterior Angle, if we so desire. At least, it makes it possible for the teacher to understand the proof, if ever called upon to explain it, and to recognize Euclid's assumption in his oft repeated proof. The Separation Postulate enables us to clarify the whole concept of angles related to two intersecting straight lines. In a number of Euclid's 'proofs', he tacitly assumed that a ray which lies in the interior of an angle of a triangle meets the opposite side of the triangle; and conversely, that the ray determined by a vertex and a point of the opposite side lies in the interior of the angle. This second statement is easy to prove on the basis of the Separation Postulate, but the first statement is difficult to prove. Because of this both statements, or at least the first, may well be stated as a Postulate without proof, but neither statement should be overlooked, if we wish to avoid the mistakes of Euclid.

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[1] See Chapter 5. A formal proof that the Plane Separation Postulate implies the Axiom of Pasch is given as Theorem 5.4.

One of the main advantages of the use of the Axioms of Linear Measure and Angular Measure is that the whole theory of congruent segments or unequal segments, and of congruent angles or unequal angles can be put upon a precise arithmetical basis independent of any notion of motion. The necessary connection between congruent segments and congruent angles is supplied as in Hilbert's Foundations by assuming the (S.A.S.) Theorem as a postulate. Whether or not you prove that (A.S.A.) Theorem and the (S.S.S.)

Theorem is a matter of how much rigor you wish to include. Emphasis is placed upon the idea of one-to-one correspondence, and the Isosceles Triangle Theorems are proved by making the triangle ABC correspond to itself, that is, triangle CBA, by making A, B, and C correspond respectively to C, B, and A. A more conventional proof is also included, but now the Axiom of Separation or some other postulate is available, if desired, to fill in the gap in the proof.

We follow Hilbert in the use of the Playfair Parallel Postulate: "Through a point not on a given line there is at most one line parallel to the given line." Euclid proved the existence of parallel lines on the basis of the Theorem of the Exterior Angle, by showing that "if a transversal intersects two lines so that the alternate interior angles are equal, the lines are parallel." It would have been simpler, after this is done, to assume its converse as the parallel postulate: "If a transversal intersects two parallel lines, then the alternate interior angles are equal." One reason for it is that this angle criterion is the property that is used as the basis for geometric constructions. For some reason not fully understood, Euclid considered what today we call the opposite and contrapositive of the Propositions he had proved, but we recognize today that these concepts need play no role in our geometry as long as we focus attention upon a theorem and its converse, and are willing to use the method of proof by contradiction when needed. The Playfair Postulate and the Transversal Theorem for Parallel lines are logically equivalent, and it is not difficult to prove either from the other.

The theory of parallel lines is usually followed by the theory of similar triangles and the related topic of proportion. The first and basic theorem to be proved may be stated as follows:

The Basic Proportionality Theorem. If a line parallel to one side of a triangle intersects the other two sides in distinct points, then it cuts off segments which are proportional to these sides.

Several avenues of approach are available. (1) Take the theorem and its converse as postulates, with or without a proof of the theorem in the commensurable case. (2) Give a proof for all cases based upon the theory of limits, or equivalently, on Eudoxes' "method of exhaustion" as presented by Euclid. Such proofs involve rather subtle properties of the real number system and are definitely not for "school-boys". However, we give such a proof here (at the end of Chapter 8) merely to show what it is like. (3) Adopt the point of view of Birkhoff (see Birkhoff and Beatley, Basic Geometry) and assume the (S.A.S.) statement for similarity as a basic postulate. (4) Follow Euclid and base the proof on the area concept. (See Euclid's Elements, Book VI, Proposition 2.) This means the development of the area concept, including the Postulates of Measurement of Area, before the discussion of similar triangles. The fundamental assumption that the area of a rectangle is its length times its breadth, a notion familiar to all students, bypasses any continuity argument and enables one to give a readily understandable proof of the basic theorem. This point of view is not common in elementary texts written in America, but it is the point of view we finally adopted.

The rest of our program is conventional, but we are in a better position to develop the theory of similar triangles and the Theorem of Pythagoras than most texts because of our Postulates of Linear Measure. The ground work for the study of circles and spheres, either synthetically or by means of cartesian coordinates in two and three dimensions has been laid. How far we go is a matter of the audience and our objectives.

We do not repeat the mistakes of Euclid.[1] We begin with undefined terms, stating explicitly what terms are undefined, and make precise Postulates about them. We do not list all the Postulates first, but begin with a few and see what can be done with them. We introduce new Postulates when they are needed. We make definitions for convenience only, and they are precise and never circumlocutions. We do not expect to present a system of minimum postulates, but make our postulates strong enough to achieve our goal of proving the theorems from the postulates in a manner that the student should be able to follow without memorizing the details of the proof. We have planned the course in geometry so that it is integrated with the students' previous knowledge, and so that it may be easily integrated with courses that follow it. We have done all this keeping in mind the spirit of Euclid's ideal, the logical derivation of geometric theorems from a set of premises completely laid down in advance.

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[1] Will mathematicians 2000 years hence point out the mistakes in our work (or, more significantly, in Hilbert's or Birkhoff's)? Undoubtedly such criticism will come in a small fraction of the time. Standards of mathematical rigor change with time, and while the present theory may be completely free from mistakes according to today's viewpoint, we may have well overlooked some fine point or made mistakes of a higher order of subtlety than those we realize were made in the past centuries. Some future geometer is sure to point these out.

# Chapter 2

## Logic

1. Sets[1]. Mathematics is often concerned with collections of objects, rather than with individuals. One frequently uses such phrases as "the integers", "the prime numbers", "the points equidistant from two given points", "the vertices of a given triangle", "the lines parallel to a given line", etc. Instead of speaking of a "collection", and "assemblage", or other such descriptive term mathematicians have pretty generally adopted the word "set", and for each of the individuals making up the set the word "element". Thus, the fourth set given above has as elements the three points which are the vertices of the given triangle. The elements of a set are said to belong to the set, and the set is said to contain its elements.

A set itself has a certain identity. A line is a set of points, but one can also consider it as an individual and talk about "a set of lines" as in the fifth example above. Much of modern mathematics is concerned with sets of sets of sets of sets $\cdots$ .

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[1] Sections 1 and 2 have been kept brief because in the geometry text only the language of sets is employed rather than set notation and set relations. A simple presentation of set theory can be found in Introduction to the Theory of Sets by Joseph Breuer, translated by Howard F. Fehr, 1958, Prentice-Hall.

If S designates a set and x one of its elements we write $x \in S$, read "x belongs to S", or "x is an element of S", or "x is in S".

In geometry we are primarily interested in sets of points, or point sets for short, although for technical purposes we must also consider sets of numbers. (See Chapter 4.) A point set is often referred to by the more familiar name of "figure".

To specify a set we must give a criterion whereby one can tell without ambiguity whether any given object is or is not an element of the set. "The set of all even integers" is well defined, but "the set of brown cows" is not unless we specify the time and agree on exactly what is meant by a brown cow.

"The set of brown cows in my (the author's) office at 5:15 P.M. on April 3, 1959 A.D." is a well defined set. It is the empty set, the set which has no elements. It may seem silly to call this a set, but remember that at one time it seemed silly to have a symbol for zero. We shall see that the empty set can serve a useful purpose.

The same set of points may be specified in two, or more, different ways. In fact some basic theorems of geometry consist of a statement to this effect; for example, "The locus of points equidistant from two given points is the perpendicular bisector of the line segment joining these two points." (The word "locus", as used in elementary geometry, is essentially synonymous with "set".) If S and T are specified sets and we write S = T we mean that S and T consist of the same elements, or in other words that "S" and "T" are just different symbols for the same set. Our use of the symbol "=" and the word "equal" will be consistent in this respect. "Equal" will mean "the same in all respects" or "identical". This is contrary to the practice in most elementary geometry books, where "equal" means different things according to the context.

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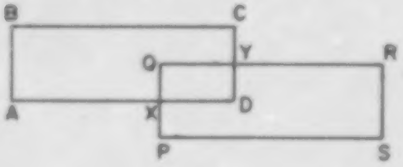
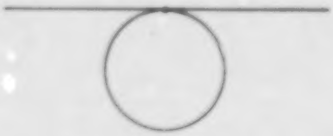
2. Relations among Sets. If a set S is entirely contained in a set T, S is called a subset of T, and we write $S \subset T$. More precisely, $S \subset T$ if $x \in T$ whenever $x \in S$.

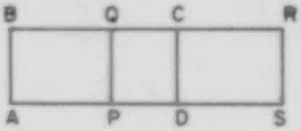
If S and T are sets, the set of elements common to S and T is called their intersection, and is designated by $S \cap T$ (read "S cap T"). In symbols, $x \in S \cap T$ provided $x \in S$ and $x \in T$. The word "intersection" is of course borrowed from geometry, where it is customarily used in precisely this sense, as in speaking of "the intersection of a plane and a sphere", etc. A less familiar use is illustrated in the adjacent figure. The intersection of the two rectangular regions ABCD and PQRS is the region XQYD. Also, according to our definition, the intersection of a circle and a tangent line is the point of contact. Finally, if two





sets have no common elements at all their intersection is the empty set. In this case we frequently use more common language and say that the two sets "do not intersect".

In contrast to the intersection of $S$ and $T$ we define their union, $S \cup T$ ("S cup T") to be the set of points in either $S$ or $T$ or both. That is, $x \in S \cup T$ provided $x \in S$ or $x \in T$ or $x \in S \cap T$. The union of region ABCD and region PQRS is region ABRS.



## Exercises

1. Consider the following point sets:

$P$ is a plane;

$C$ is a circle in $P$;

$L$ is a line intersecting $C$ in two distinct points $X$ and $Y$;

$S$ is the line segment consisting of $X$, $Y$, and all points of $L$ between $X$ and $Y$;

$Q$ is the set of points inside $C$;

$W$ is the set consisting of the two elements $X$ and $Y$.

(a) Which of the following are true?

(i) $C \subset P$, (ii) $Q \subset P$, (iii) $Q \subset C$, (iv) $C \subset Q$, (v) $S \subset Q$, (vi) $W \subset S$, (vii) $W \subset Q$, (viii) $S \subset L$.

Write all the true inclusion relations among these six sets.

(b) Which of the following are true?

(i) $W = C \cap L$, (ii) $S = Q \cap L$, (iii) $W = S \cap L$, (iv) $W = S \cap C$, (v) $S = L \cap (Q \cup C)$, (vi) $S = (Q \cap L) \cup W$, (vii) $Q \cap S \subset C$, (viii) $Q \cap C \subset S$.

(c) Describe in geometric terms each of the following sets:

(i) $Q \cap S$, (ii) $C \cup L$, (iii) $Q \cup C$.

2. If $S$ is a set, is $S \subset S$ ever true? always true? sometimes true?

3. Can you assign a reasonable meaning to $S - T$ in all cases? in some cases?

4. Give definitions for the intersection and the union of any number of sets.

5. (a) Show that $(S \cap T) \subset T$ and $T \subset (S \cup T)$.

(b) Show that $S \cap T = T$ is equivalent to $T \subset S$.

(c) Show that $S \cup T = T$ is equivalent to $S \subset T$.

6. Show that $S \cup (T \cap R) = (S \cup T) \cap (S \cup R)$, and that $S \cap (T \cup R) = (S \cap T) \cup (S \cap R)$.

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3. Correspondences. It is sometimes necessary to consider a certain type of relationship between two sets. Let S and T be sets, and suppose there is a well-defined rule which associates certain pairs of elements, the first element of the pair being from S and the second from T. Such a rule is called a correspondence between S and T.

Example 1. Let S be the set of all points of a given plane and T the set of all circles lying in that plane. We associate a point P with a circle C if P is the center of C. This is a one-to-many correspondence; each circle corresponds to exactly one point, its center, but each point corresponds to many circles.

Example 2. With S and T as above let a point and a circle correspond if the point lies on the circle. This is obviously a many-to-many correspondence.

Example 3. Let S be as before but let T consist only of those circles with a radius of one inch. Then the correspondence of Example 1 is one-to-one.

The last example illustrates the most important type of correspondence. A correspondence between S and T is said to be one-to-one if each element of S is associated with exactly one element of T, and each element of T with exactly one element of S. Such a "pairing off" of the elements of the two sets often enables one to carry over to the second set some of the properties of the first.

## Exercises

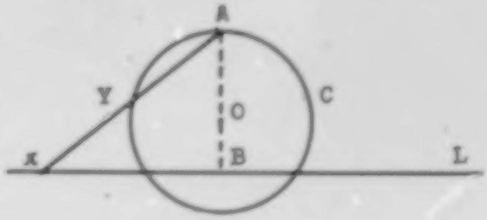
1. Discuss the obvious correspondence between book covers and colors.

2. Is it possible for a set to have a correspondence with itself? with a proper subset[1] of itself? could such a correspondence be one-to-one?

3. If $x$ and $y$ represent integers discuss the correspondences:

(i) $x \longleftrightarrow -x$,
(ii) $x \longleftrightarrow x^2$,
(iii) $x \longleftrightarrow 3x$,
(iv) $x \longleftrightarrow y$ if $x + y = 7$,
(v) $x \longleftrightarrow y$ if $x^2 = y^2$,
(vi) $x \longleftrightarrow y$ if $x^2 = y^3$.

4. In this figure O is the center of the circle C, AB is perpendicular to line L, and $X \longleftrightarrow Y$ is a correspondence between points X of L and points Y of C. Is this correspondence one-to-one?



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[1] By definition, every set is a subset of itself. S is a proper subset of T if $S \subset T$ but $S \neq T$.

4. Sentences. In all mathematics and particularly in geometry we are interested in drawing conclusions from explicitly stated hypotheses in accordance with assumed laws of logic. To appreciate this statement fully, it will be necessary to discuss what we mean by a number of terms used in geometry and to state the assumed laws of logic. To communicate ideas we use words (or symbols) which form sentences.[1] We confine our attention to such statements or sentences which we assume are either true or false, but not both, and are not meaningless but have content.[2] If a statement is written to which this assumption does not apply (and there are such statements), we exclude it by agreement from our discourse. The assumptions that the statement must be true or false, but not both, are often referred to as the laws of Contradiction and the Excluded Middle of Aristotelian logic. The

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[1] Descriptive definitions of a sentence to be found in a dictionary might be these: (1) A related group of words expressing a complete thought. (2) A verbal expression of an idea which associates a person, thing, or quality, expressed in the subject, with an action, state, or condition, expressed in the predicate. We will use the words statement and sentence as synonymous.

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[2] In order to avoid philosophical or semantic discussions of the terms true and false, it is often found convenient to use the words "valid" and "invalid".

assumption that the statement must have content, and thus not be meaningless, means that we must be able to give at least one, and possibly more than one, interpretation to all the terms used. We may not know whether the statement is true or false, but we must be willing to accept the fact that it is either true or false but not both. We are willing to accept the sentence: "$4 = 2 \times 2$" as true, the sentence: "$4 = 3$" as false, and the sentence: "4 = a parallelogram" as nonsense. On the other hand, we do not know whether the sentence: "The decimal expansion of the irrational number $\pi$ contains ten consecutive 7's" is true or false, but we are willing to accept it as being true or false, but not both.[1]

5. Descriptive Definitions. To communicate ideas we use words (or symbols) and we must have some idea about what they mean. We usually define certain terms by means of other words, and we should have some preliminary notion of what constitutes a (good) definition. Some definitions are purely descriptive and informal, and by citing special instances, giving illustrations or drawing a picture, give some meaning and understanding to the term to be defined. But such definitions cannot be used as a logical basis for the development of geometry. Perhaps such a definition explains

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[1] The fact that this is an assumption is emphasized by the existence of schools of thought that reject this postulate, especially the law of the Excluded Middle.

the term to be defined by means of other words already defined; these latter terms might be defined by means of still other simpler (?) terms, and so on. But the process cannot go on forever. Eventually, if we wish to avoid circumlocutions which are not logically acceptable, we must arrive at a few terms that are not defined, but for which we have some, perhaps more than one, interpretation. If we draw a picture or use a physical model to illustrate the term defined, they are only special instances, and there may be other models that could fit just as well. If we wish to define curve passing through two points, I am sure you could draw many pictures. The above type of descriptive definition is often found in standard dictionaries, and the fact that a single term may have many interpretations is well illustrated in such dictionaries.

The first thing we must acknowledge for any logical discourse is that there must be some terms that are not defined. We want such terms to have some interpretations, and indeed, perhaps more than one. We do not want them to be without any interpretation or to be meaningless. However, logical deduction must be independent of the particular interpretation which might be attached to the undefined terms.

In geometry we may consider such entities as point, line, plane, space as undefined. Some of the relationships such as on, contains, equal, congruent, greater than, between, separate, length, etc., and such operations as addition and multiplication (and many others) may be left undefined. There are certain logical terms as set, and, or, there exist, all, at least, true, etc., some of which may be left undefined. The set of terms left undefined is somewhat arbitrary and is determined by the objective of the discourse. Different sets of terms may even be used for the same objective. Other terms mentioned are then defined by means of the undefined terms.

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6. Postulates. After the set of terms that remain undefined is selected, we make statements (sentences) about these terms. We accept these statements to be true without proof in the sense that these statements form part of our hypotheses from which all other conclusions are logically proved. They impose some conditions upon the undefined terms, so these terms cannot be considered to be absolutely arbitrary. By means of these hypotheses we delimit that part of the universe we are talking about; we are talking about any system of things possessing the properties expressed in these assumptions. The more such assumptions we make, the more limitations we impose, and the more difficult it becomes to realize a model from our limited experiences.

Such assumptions are variously called postulates, axioms, principles, or agreements. We shall make no distinction between such names and usually use the word postulate, because axiom has too often been associated with the idea of "self-evident truth". In our geometry there are no self-evident truths; the postulates are assumed to be true. Postulates, of course, are not made up at random; they are suggested by fundamental properties of physical space, just as the undefined terms, point, line and plane are suggested by physical objects. Naturally, the particular postulates we use in elementary geometry are based on our experience with physical models, and are usually accepted without question on the basis of such experience. That is intended, for we want elementary geometry to be an idealization of certain aspects of experience. The physical models suggest, by idealization, certain properties that would presumably be possessed by certain highly idealized substitutes for the physical objects. Among the many properties that we suspect these idealized substitutes of having, some can readily be deduced logically from the others and would be omitted from the postulate system; the remainder of these properties could then be taken as a set of postulates. The postulates used in elementary geometry are based on empirical considerations; they are to be regarded, however, as independent of such empirical considerations. Indeed they might have more than one empirical interpretation, or, as we say, we may present more than one model which can be used to give an empirical interpretation to the same postulate system. In this way we hope, by proof, to make discoveries without explicit experience, or to use facts gained by proof as a check on our experience and vice-versa.

As an illustration, let us take the terms point, line, and contains as undefined, and assume that the other (logical) terms have meaning.

Postulate 1. Given two distinct points, there exists one and only one line which contains them.

We might think of a model in which the term "line" has the interpretation of a stretched string and 'contains' suggests the equivalent idea of 'passing through'. If this is the only postulate made, there are other models which also apply. We use an illustration from elementary analytic geometry. Suppose the 'two points' are interpreted as the ordered number pairs (0,0) and (1,1), the "line" as the equation $y = x$, and 'contains' as 'the number pairs satisfy the equation'. You can recognize, with this interpretation, the sentence, called Postulate 1, is true. However, if the word "line" is interpreted as the equation $y = ax^2$, you may also verify (and, hence, accept as true) the sentence:

"Given the distinct points (0,0) and (1,1) there exists one and only one parabola of the form $y = ax^2$ which contains them."

Here the model used for the word "line" is the 'parabola of the form $y = ax^2$'. If no further postulates about 'point', 'line' and 'contains' are made, other interpretations are also available:

"Given two different boys, there exists one and only one committee of two upon which they serve."

Of course, in elementary geometry, other postulates are made, so that all of these interpretations are not then simultaneously valid.

7. Explicit Definitions. Hereafter, the word definition will refer to explicit definition to distinguish it from descriptive definition as used in Section 5.

(1) An explicit definition of a term is a characterization of the term by means of its attributes, properties, or relations that distinguish it from all other words that have different meanings.

(2) The definition must be reversible, in the sense that the distinguishing properties must be both necessary and sufficient to yield the term defined.

(3) To be used in a logical discourse, it must use only those terms which have previously been explicitly defined or accepted as undefined but limited by explicitly stated postulates.

Let us illustrate these ideas by several examples. From this point of view the definition of line segment found in Euclid's Elements and in practically all of his American imitators is unsatisfactory.

Definition. A line segment $\overline{AB}$ is the set of all points "between" two distinct points A and B.

First, a line segment is a set of points, and this set is distinguished from all other sets of points, in that it contains all points "between" two distinct points; second, the fact that we call the sentence a Definition implies that all points "between" the given points belong to the set. Third, this is an acceptable definition for a logically self-contained system if and only if the term "between" has been defined before line segment is defined, or "between" is taken as undefined subject to a set of postulates, before line segment is defined. It is not an acceptable definition if "between" is used in an intuitive sense or merely illustrated by a figure. Otherwise you might be tempted to say: "A point is between two distinct points A and B, if it is contained in the line segment $\overline{AB}$". Unfortunately such circularities in definitions are much too common in American texts.

The term defined is fundamentally an abbreviation for a much longer group of phrases, but it could always be replaced by a statement of its distinguishing properties. However, definitions are very convenient and important in helping us think accurately and concisely, and help us avoid using words carelessly and with muddled meanings.

## Exercises

1. Taking the undefined terms given at the end of Section 5, examine each of the following statements to see if it is an acceptable definition. If not, see if you can modify it to make it one.

(a) A straight line is one which lies evenly between all its points.

(b) A line segment is the set of points contained in a given line and lying between two given points of the line.

(c) The angle between two lines is the amount of turning required to make one line coincide with the other.

(d) Parallel lines are two lines contained in one plane and which do not contain a common point.

(e) A circle is a closed plane curve, all points of which are the same distance from a fixed point in that plane.

(f) The distance from a point to a line is the shortest path from the point to the line.

2. The treatment of sets in Sections 1 and 2 is intuitive, not logical. Give a logical treatment, selecting and giving suitable definitions for subset, intersection, and union.

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8. Theorems. In accordance with the agreement of Section 4, the statements or sentences we are to consider in geometry are either true or false, but not both. When we call a statement a Definition, it is to be understood that the statement is true, without explicitly saying so. When we call a statement a Postulate (or any equivalent terms), we understand it is true by assumption. There are other statements, of which, without further information, we cannot tell whether they are true or false. Examples of such sentences, some simple and some compound, follow:

(1) Two line segments are congruent.

(2) The triangle is isosceles.

(3) The lines are perpendicular.

(4) The point $P$ is not on the line $u$.

(5) $a$ and $b$ stand for real numbers and $a = b$.

(6) If $a, b, c$ stand for real numbers, and if $a = b$ and $c = d$, then $a - c = b - d$.

(7) If two sides and the included angle of one triangle are respectively congruent to two sides and the included angle of a second triangle, the remaining sides are congruent.

In examples (1) - (5), we do not know whether they are true or false, and cannot determine which unless more information is available about the objects mentioned. Examples (6) and (7) are written in the form of compound hypothetical statements called conditionals. If we accept them as postulates they are true by assumption. On the other hand, it might be possible to prove them on the basis of other postulates.

A Theorem is a statement which can be proved to be true in accordance with the stipulated laws of logical deduction discussed in Section 9 on the basis of the accepted postulate system.

Theorems of geometry are written in two different forms: they may be written as a direct simple sentence or as hypothetical compound sentences (conditionals). We are justified in calling them Theorems (Euclid used the equivalent term, Propositions) only after they have been proved to be true. The statement corresponding to any Theorem may be written as a hypothetical compound sentence (called a conditional) in the form

If p, then q,

where p and q are symbols standing for simpler sentences. We also say p implies q, and use the symbolic form $p \Longrightarrow q$. We are justified in calling it a Theorem when (If p, then q) is true, where the truth is established by logical means discussed below. Before considering the methods of proof, let us illustrate the ideas above by examples which are usually stated as Theorems of Euclidean plane geometry.

1. Two distinct lines intersect in at most one point.

2. One and only one perpendicular can be drawn to a line from a point not on it.

3. The sum of the measures of the angles of a triangle is $180^\circ$.

4. The diagonals of a rectangle have the same length.

5. The diagonals of a square are perpendicular.

Before these are proved, we are merely justified in calling them sentences. These sentences can be translated into the form: "If p, then q." The sentence p is called the hypothesis; it is the statement which is assumed true, as indicated by the word "If". It represents the facts which we think of as given. The sentence q is called the conclusion; it is the statement which is to be proved true, as indicated by the word "then". It represents the facts which are to be obtained by proof. After the compound statement "If p, then q" is shown to be true, we call it a Theorem. The complete sentence; p implies q is true, is called an implication.

Each of the five sentences given above are now translated into alternative forms.

1a. If m and n are two distinct lines, then they have at most one point in common.

Statement 1 or 1a is a Theorem of plane geometry after we prove it, not merely because we state it.

As a related example, we could write the following two equivalent forms of a sentence:

6. Two distinct lines lying in the same plane have a point in common.

6a. If m and n are two distinct lines in the same plane, then they have a point in common.

Actually the sentence 6 (or 6a) is not a Theorem of plane Euclidean geometry in that it is possible to prove (or it is often assumed) that there are distinct lines in the Euclidean plane that have no point in common. That is, sentence 6 (or 6a) is false in Euclidean geometry, but there are geometries in which it is true.

2a. If m is a line and C is a point not on it, then there is one and only one line which passes through C and which is perpendicular to m.

3a. If $\alpha, \beta, \gamma$ are the measures of the angles of a triangle, then $\alpha + \beta + \gamma = 180^\circ$.

4a. If a given quadrilateral is a rectangle, then its diagonals have the same length.

5a. If a given quadrilateral is a square, then its diagonals are perpendicular.

Statements 1 - 5 (or 1a - 5a) are Theorems because they are capable of proof. However, the following statements 6 (or 6a) and 7 (or 7a) are not Theorems, because it is possible to prove they are not true. By means of a rhombus with unequal but perpendicular diagonals, it is possible to show statement 7 is false.

7. The quadrilateral whose diagonals are perpendicular is a square.

7a. If the diagonals of a quadrilateral are perpendicular, the quadrilateral is a square.

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9. Methods of Proof. In order to simplify the discussion, we usually write all hypothetical sentences (or conditionals) in the form: "If p, then q" and discuss the laws of logic by means of which the validity or truth of the conditional is established, and thus justify calling the sentence a Theorem. The intent to prove the statement is true is indicated by writing: "If p is true, then q is true" or

"Given p is true, prove q is true."

The basic method of proof is a direct application of the following rule (assumption) of inference:

Rule of Inference: If "p" is true and if the conditional "If p, then q" is true, then "q" is true.

In the simplest proof, p is the hypothesis which is known to be true or which we assume is true, and the conditional: (If p, then q) will be true if it is the statement of a postulate, definition, or a previously proved Theorem. The rule of inference then states that the conclusion q is also true.

The conditional "if p, then q" is commonly stated in mathematical terminology in the following six ways:

If p, then q,

p implies q,

q if p,

p only if q,

q is a necessary condition for p,

p is a sufficient condition for q.

At this point it is also worth while clarifying the statement p if and only if q, called a biconditional. This statement is simply a brief way stating [(if p, then q) and (if q, then p)]. In other words a theorem expressed in the form "p is true if and only if q is true" really contains two statements to be proved, namely, "if p is true, then q is true" and "if q is true, then p is true". Some other ways of stating "p if and only if q" are:

q if and only if p,

p is equivalent to q,

a necessary and sufficient condition for p is q,

a necessary and sufficient condition for q is p.

Another logical term we have to discuss is that of the converse of a conditional statement. We define the converse of the statement "if p, then q" to be the statement "if q, then p".

We have agreed that our sentences are either true or false. The negation of a sentence "p is true" is the sentence "p is false" which we shall sometimes express also in the form "not p".

So far all of our discussions has centered around the interpretation of a single conditional statement "if p, then q". The applicability of the Rule of Inference is enormously extended by the next rule of logical reasoning.

Rule of the Syllogism (or Rule of Transitivity of Implication). If p implies q and q implies r, then p implies r.

In other words, suppose the hypothesis of a theorem is p, and the conclusion r. Suppose among our postulates and definitions we can find "p implies q" and "q implies r". Applying the Rule of the Syllogism we can state "p implies r", and because the hypothesis of our theorem is p, we can apply the Rule of Inference to conclude r, and the theorem is proved.

The final method of proof which we shall describe is the method of Indirect Proof, or proof by contradiction, as it is sometimes called. First of all, we define a contradiction to be a sentence of the form "p and not p; that is, "p is true and not p is true", or, "p is true and p is false". A fundamental assumption concerning our logic is that every contradiction is false.

Rule of Indirect Proof. A conclusion q is true if (not q) implies a contradiction.

In the simplest case, to establish the statement "If p, then q", we assume the negation of q. If we can derive, using the rules of logic, the statement "not p", then q is true. More generally, if from the hypothesis "p" and the assumption "not q" we can then derive, using the rules of logic, a statement of the form "r and not r", that is derive both "r" and "not r", we can conclude that our original hypothesis that q is false is invalid, and, hence, that q is true. There is no general rule which tells us how to find the contradictory statement "r and not r", so that the method of indirect proof sometimes is a more difficult method to apply than the other rules of inference. We shall find many cases, however, where it seems to get at the heart of the matter in short order.

From the simplest case of the Rule of Indirect Proof, it is possible to derive the following:

The conditional "if p, then q" is logically equivalent to the statement "if not q, then not p", called the contra-positive of the original conditional.

The converse "if q, then p", is logically equivalent to the statement "if not p, then not q", called the opposite of the original conditional.

This logical equivalence indicates that to establish the biconditional it is sufficient to discuss the conditional and its converse. At times it may be more convenient to use the opposite instead of the converse.

So far all of our attention has been concentrated on methods of proof. It is worthwhile making a remark conerning methods of disproof. Far too little emphasis is given in the usual high school course to the possibility of having the students discover theorems for themselves. After all, this is the way mathematics is done; a theorem first has to be guessed, before it can be proved. [In this connection the reader is urged to read in G. Polya's books "How to Solve It" and "Mathematics and Plausible Reasoning", both published by the Princeton University Press.] Assuming that we are trying to guess some theorems, we shall arrive at a list of sentences to test. Some we may be able to prove; others will resist our attempts at a proof, and we may be led to entertain the possibility that a statement is false. How do we show that the statement is false? Consider the following statement.

For every triangle with sides of length $a$, $b$, $c$ it is true that $a^3 = b^3 + c^3$.

We are not impressed with the statement; obviously, it is false. Why? Well, we know a triangle can have sides of lengths 3, 4, and 5, and it is false that $5^3 = 4^3 + 3^3$. Such an example shows that the statement is false. A general statement, if true, will remain true in every special case to which it can be applied. Therefore a general statement is false if there exists one special case in which the statement is false. This special case is called a counter-example to the statement. For a creative student of mathematics, it is just as important to be able to find counter-examples to explode wrong guesses as it is to find proofs of statements that will turn out to be theorems.

## Exercises

1. Find a counter-example to disprove the statement "If $n$ is a positive integer then $n^2 + n + 11$ is a prime number."

2. Using the simplest case of the Rule of Indirect Proof, prove that (1) "if $p$, then $q$" implies "if not $q$, then not $p$", and (2) "if not $q$, then not $p$" implies "if $p$, then $q$".

3. Prove the logical equivalence of the converse and the opposite of a given conditional.

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## Chapter 3

## Points, Lines, and Planes

1. Introduction. This chapter contains a list of undefined terms, and the first definitions and postulates upon which our entire development of geometry is based. The language of sets introduced in Chapter 2 will be freely used from the beginning.

It should be emphasized that there are other sets of postulates equally satisfactory from a logical point of view to those we have adopted. Those in the present chapter, however, contain no surprises, and we shall not attempt to motivate them. The postulates given in Chapters 4 and 5, however, are not commonly used in high school geometry courses, and we have included rather full discussions of them in the Introductions to Chapters 4 and 5.

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2. Definitions and Postulates. We begin with a list of undefined terms.

Undefined Terms.

Point.

Line: a set of points otherwise undefined.

Plane: a set of points otherwise undefined.

Definition. The set of all points is called space.

Next, we consider various statements concerning points, lines, and planes. Because lines and planes are sets of points, we should use the terminology "$P$ is an element of $\ell$" or "$P \in \ell$" for the statement that a point $P$ belongs to the line $\ell$. Instead we shall often say $P$ *lies on* $\ell$, or $\ell$ *passes through* $P$, etc.

*Definition*. Points lying on one line are said to be *collinear*. Points lying in one plane are said to be *coplanar*.

The first postulate guarantees that our geometry contains enough points to be interesting.

*Postulate 1*.

(a) Every line contains at least two distinct points.

(b) Every plane contains at least three distinct non-collinear points.

(c) Space contains at least four distinct non-coplanar points.

*Postulate 2*. Given two distinct points, there exists one and only one line containing them.

*Notation*. We shall denote the line containing the distinct points $A$ and $B$ by $\overleftrightarrow{AB}$.

*Postulate 3*. Given three distinct non-collinear points, there is one and only one plane containing them.

*Postulate 4*. If two distinct points lie in a plane, the line containing these points lies in the plane.

What does Postulate 4 assert? Given two points A and B which lie on a plane p, Postulate 2 asserts that A and B lie on a unique line $\overleftrightarrow{AB}$. Postulate 4 then states that the line $\overleftrightarrow{AB}$ is a subset of p, in other words, every point on $\overleftrightarrow{AB}$ is also on p.

*Postulate 5*. If two distinct planes intersect, their intersection is a line.

3. *Some Basic Theorems*. A remark about the numbering of statements is appropriate. Theorem 3.1 means the first theorem of Chapter 3; Theorem 4.2 means the second theorem of Chapter 4, etc. Results of lesser importance are numbered according to the section of their chapter, e.g. (4.2) of Chapter 4 means the second result in Section 4 of Chapter 4.

*Theorem 3.1*. Two distinct lines have at most one point in common.

Proof: First of all we must interpret the statement of the theorem. We are given two distinct lines $\ell$ and $\ell'$. Their intersection consists either of no points, one point, or more than one point. Our task is to show that the third possibility cannot occur. Here in our first theorem we know of no way to proceed except by the method of indirect proof! How does the method look in this case? We have a statement p: "two distinct lines have at most one point in common." To make an indirect proof we must show that (not p) implies a contradiction. What is (not p)? It is the statement that for some pair of distinct lines $\ell$ and $\ell'$, the intersection of $\ell$ and $\ell'$ contains more than one point. Therefore the intersection of $\ell$ and $\ell'$ must contain two distinct points A and B. To say that A and B belong to the intersection of $\ell$ and $\ell'$ means that A and B lie on the line $\ell$ and that A and B lie on the line $\ell'$. Applying Postulate 2, which states that through two distinct points passes one and only one line, we conclude that the lines $\ell$ and $\ell'$ coincide. We have reached a contradiction, namely, the statement

q and (not q),

where q is the assertion "$\ell$ and $\ell'$ are distinct lines". Summing up the whole argument, we have shown that (not p) implies the contradiction q and (not q), and by the Rule of Indirect Proof we conclude that Theorem 3.1 is true.

Theorem 3.2. If a line intersects a plane not containing it, the intersection is a single point.

Proof: This time the Theorem takes the form of a conditional statement. What is the hypothesis? We are given a line $\ell$, and a plane p not containing $\ell$, such that, p and $\ell$ do have a point or points in common. We are trying to show that they have exactly one point in common. Again we use an indirect proof. This time, the statement of the theorem is false if there exists a line $\ell$ which intersects the plane p in more than one point, but such that $\ell$ is not contained in p. Let A and B be distinct points which lie on both $\ell$ and p. By Postulate 2, there exists one and only one line containing A and B, and because $\ell$ contains A and B this line is $\ell$, or more briefly, $\ell = \overleftrightarrow{AB}$. By Postulate 4, the line $\overleftrightarrow{AB}$ lies in the plane p. We have derived the contradiction

$$q \text{ and } (\text{not } q),$$

where this time q is the statement "$\ell$ is contained in p". Again by the Rule of Indirect Proof, we can assert the truth of our theorem.

Theorem 3.3. Given a line and a point not on the line, there is one and only one plane containing the line and the point.

Proof: Again we have a conditional statement to prove, and this time we shall spare the reader another indirect proof. We are given a point P and a line $\ell$ such that P does not lie on $\ell$. By Postulate 1, there exists two distinct points A and B on $\ell$. Then we can assert that A, B, and P are non-collinear. [The reader is asked to supply the proof of this statement.] Applying Postulate 3, there exists one and only one plane p containing A, B, and P. Because of Postulate 2, $\ell = \overleftrightarrow{AB}$ and from Postulate 4, we conclude that $\ell$ is contained in p. We have proved that the plane p contains P and the line $\ell$. Because any other plane containing $\ell$ and P must contain A, B, and P, the fact that p is the unique plane containing A, B, and P implies that p is also the unique plane containing P and $\ell$, and Theorem 3.3 is proved.

Theorem 3.4. Given two distinct lines with a point in common, there is one and only one plane containing them.

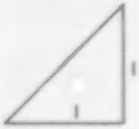
Proof: This time we have to prove a conditional statement with hypothesis that we are given two distinct lines $\ell$ and $\ell'$ which intersect in a point P. By Postulate 1, the line $\ell'$ contains a point P' different from P. By Theorem 3.1, P' does not lie on $\ell$. By Theorem 3.3, P' and $\ell$ are contained in one and only one plane p. Because $\ell'$ is the unique line containing P and P', Postulate 4 guarantees that the plane p contains also the line $\ell'$. Finally, any other plane containing $\ell$ and $\ell'$ contains P' and $\ell$, and because p is the unique plane containing P' and $\ell$, we conclude that p is also the unique plane containing $\ell$ and $\ell'$. This completes the proof of Theorem 3.4.

## Chapter 4

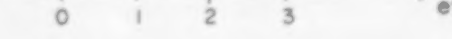
# Real Numbers and the Ruler Axiom

1. Introductory Discussion. Most of the geometrical questions that occur to us in every day life involve the notion of length. It is interesting to learn that the Greek geometers made very little use of the concept of length, and for a very good reason. At the time Greek geometry was invented, the numbers which we take for granted nowadays were understood only in a most rudimentary and imperfect way. In particular, the distinction between rational and irrational numbers was a source of great mystery to the early geometers, who realized that the hypotenuse of a right triangle both of whose legs are one unit long must have a length, but that

(1)



this length could not be compared in a simple way with the unit length of each side. In more detail, they were able to measure lengths which were whole number multiples of unit length, and which could be constructed from the unit length with a compass



etc.

They defined two line segments

and

to be commensurable if it was possible to form a segment $mA$ whose length was some whole number $m$ times the length of $A$, and to form another segment $nB$ whose length was some other whole number $n$ times the length of $B$, in such a way that the segments $mA$ and $nB$ were congruent. More briefly we would say today that the length of $A$ is a rational multiple of the length of $B$, the constant of proportionality being the rational number $\frac{m}{n}$. Then the Greek geometers made the discovery that the hypotenuse of the right triangle in Figure (1) was incommensurable with either of the sides. Once again, we can state this result briefly today by saying that $\sqrt{2}$ is not a rational number.

We should also recall that the possibility of translating a problem involving lengths to algebraic equations meant little to the Greek geometers. In their time the simplest algebraic equations were regarded as difficult. For example, the reader may consult the World of Mathematics (vol. I, p. 197) to see the famous cattle problem of Archimedes. It boils down to a system of linear equations, with large coefficients and large answers to be sure, but still a problem that a bright high school student could do with ease if he had the patience to write down such tremendous numbers. The point is that for the Greeks these problems were really hard since they had no workable notation for the numbers, nor did they have the methods of algebra available to solve systems of linear equations.

With this background, it should not take much persuasion to convince ourselves that our approach to those parts of geometry involving the notions of length can and should improve upon the way these things were done two thousand years ago. Specifically, we are going to assume familiarity with the rational numbers and real numbers, and with the elementary tools of algebra. All this is applied to geometry by means of the Ruler Postulate, to be given at the end of this chapter. The idea expressed precisely in the postulate is that we are given once and for all a ruler, with a fixed unit of length, and with the property that at each point on its edge we have a mark corresponding to exactly one real number, and that every real number has its mark on the ruler (obviously this is not a ruler to be purchased in any hardware store). Then our postulate asserts that for every line $L$ in our geometry, and every pair of points $P$ and $Q$ on $L$, there is defined a real number which we shall call the distance between $P$ and $Q$, and which is measured in the following way: Our ruler is placed in such a way that its marked edge coincides with the line $L$; then opposite the points $P$ and $Q$ will lie marks which correspond to real numbers $p$ and $q$. Then the distance between $P$ and $Q$ is given by the Ruler Postulate to be either $p - q$ or $q - p$, whichever one of these numbers is positive.

For example, we might lay down the ruler in such a way that at $P$ we read $3$ and at $Q$ we read $5\frac{1}{3}$. Then the distance between $P$ and $Q$ is

$$5\frac{1}{3} - 3 = 2\frac{1}{3}.$$

On the other hand we might lay down the ruler again in such a way that at $P$ we read $1$ and at $Q$ we read $-1\frac{1}{3}$. The distance this time is $1 - (-1\frac{1}{3}) = 2\frac{1}{3}$.

Besides having the advantage of its identification with the familiar operation of measurement, the Ruler Postulate has the advantage of making it possible at an early stage to translate all problems about distance into problems about real numbers, which we shall often be able to solve by the methods of algebra.

The purpose of this chapter is to give first of all a review of those aspects of the real numbers which will be important for us, such as the properties of inequalities and absolute value, and finally to give a precise statement of the Ruler Postulate.

## A Word on the Organization of §4.2 - 4.5

A fairly complete presentation of the real number system is included, more, in fact, than can or should be covered as background for the geometry course. Nevertheless the Ruler Postulate has hidden in it, so to speak, all the properties of the real number system, and although these properties are not exploited fully until the chapter on analytic geometry, it seems to be a good idea to sketch out the properties of the number system rather fully.

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2. The Real Number System. In this section and the two succeeding ones, there will be exercises for the reader at the end of the sections. Throughout this book there will also be a number of results which will be listed with a *, for example, (2.1)*, Theorem 4.5*, etc. Proofs of these results are omitted, and the reader is invited to prove them himself. Very little benefit is to be gained from a study of the number system unless strenuous efforts are made by the reader to discover for himself proofs of the starred results, as well as solutions for the exercises.

Before starting, we make a remark on the terminology: "real number" is a technical term, and the real numbers are described exactly by the properties we assume as axioms concerning them. The fact that there are also complex or imaginary numbers should not lead the reader to believe that one sort of number is any more or less mysterious or more or less down-to-earth than the other. It happens that for elementary geometry, it is unnecessary to consider the system of complex numbers.

Everyone is familiar with at least one intuitive description of the real numbers. For example, the real numbers may be described as the collection of rational numbers $\frac{a}{b}$, where a and b are integers, together with all "numbers" which can be approximated arbitrarily closely by rational numbers. Or they may be described as all numbers represented by writing finitely many digits (preceded by a + or - ), then a decimal point, and then an unending sequence of digits. Still another approach is to view them as labels for the points on a line. This is not the place to explore the connections among these ideas; we seek a precise and usable description of the real numbers to apply in setting up our geometry. To accomplish this, we proceed exactly as in geometry by giving a set of axioms for the real numbers. The axioms may be put into three groups; first the algebraic axioms for the real numbers, which can be summarized today in the assertion that the real numbers form a field; then the order axioms which enable us to discuss the size of a real number; and finally the completeness axiom which guarantees that enough real numbers exist for us to do business, for example, that there exists a real number $a$ such that $a^2 = 2$. We shall organize the material as follows: First we shall give the algebra axioms, and discuss their consequences; then we give the order axioms and discuss their consequences, and finally we give the completeness axiom.

First of all we make a remark about sets and the notion of equality. The real number system is going to be defined as a set of objects, and these objects will be denoted by symbols $\{a, b, c, \ldots, 0, 1, \ldots\}$. The symbol representing an object can be thought of as the name of the object. We shall assume that with this set we are given a means of distinguishing whether two objects are different or not. In other words, given an object with name $a$ and an object with name $a'$ we assume that exactly one of two possibilities holds:

(i) The "two" objects are really the same; in this case we write $a = a'$ (read "$a$ equals $a'$", or "$a$ is equal to $a'$"). Example: One object is "Abraham Lincoln", and one is "The 14th President of the United States".

(ii) The objects are not the same; in this case we write $a \neq a'$ (read "a is not equal to a'").

In other words, when we have assigned the same object different names a and a', we indicate this fact by writing $a = a'$.

[In the SMSG geometry text the word "equal" will be used only in this sense. The statement $\Delta\ ABC = \Delta\ XYZ$, for example, will mean that we are simply dealing with two different notations for the same triangle. The careless use of "equal" in most geometry books, to mean "having the same length" in some cases, "having the same area" in others, etc. is avoided. If we want to say that $\Delta\ ABC$ and $\Delta\ XYZ$ have the same area we say this; or we can say that their areas are equal; or we can define a new term and call them "equivalent" or "equi-areal".]

We assume that the "equals" relation has the following properties.

(i) $a = a$ (in other words, we cannot use the same symbol to stand for different objects).

(ii) if $a = b$ then $b = a$.

(iii) if $a = b$ and $b = c$, then $a = c$ (this is a precise statement of the notion that things equal to the same thing are equal to each other).

Finally we introduce a rule which will govern the use of the equal sign in our later discussion.

(SP) Substitution Principle. The real number system will be a set of objects, and it will also be possible to form combinations of real numbers by adding, subtracting, multiplying, dividing, etc., for example, we may have expressions like

$$\frac{[a + (b - c)]d}{2}\left[\frac{\sqrt{a - 1}}{3}(a^2 + b^2 - cd)\right].$$

Such an expression involving symbols for the real numbers as well as the signs $+$, $-$, $\cdot$, $\sqrt{\ }$, etc. will be called a formula provided that it has been put together in such a way that it represents a real number. The substitution principle asserts that if $a$ occurs in any formula whatsoever, and if $b = a$, then $b$ may be substituted for $a$ wherever $a$ occurs in the formula, and the resulting formulas are equal in the sense that they represent the same real number.

As an example of how this is used, suppose we have to solve the equation

$$x - 1 = 0.$$

Then usually we say "add one to both sides". This operation is justified by the substitution principle, for in the formula

$$(x - 1) + 1$$

we may substitute for $x - 1$ the symbol $0$, and obtain

$$(x - 1) + 1 = 0 + 1$$

and from this as usual we obtain

$$x = 1.$$

Exercise: Show how Euclid's first seven axioms follow from the substitution principle.

*Definition*. The *real number system* is a set of objects called real numbers, and denoted by symbols $a, b, c, \ldots, 0, 1, 2, \ldots$, etc., which satisfies the algebra axioms, the order axioms, and the completeness axiom.

*Algebra Axioms*.[1] For any pair $a, b$ of real numbers, there is defined a unique real number $a + b$, called the *sum* of $a$ and $b$, and a unique real number $a \cdot b$, called the *product* of $a$ and $b$, such that the following axioms are valid.

(F.1) $a + b = b + a$, $ab = ba$ (commutative laws)

(F.2) $(a+b)+c = a+(b+c)$, $(ab)c = a(bc)$ (associative laws)

(F.3) $a(b + c) = ab + ac$ (distributive law)

(F.4) There is a real number $0$, such that $a + 0 = 0 + a = a$ for all real numbers $a$.

(F.5) There is a real number $1 \neq 0$, such that $1 \cdot a = a \cdot 1 = a$ for all real numbers $a$.

(F.6) For each real number $a$, there exists a real number $-a$ (read "minus $a$"), such that

$$(-a) + a = a + (-a) = 0.$$

(F.7) For each real number $a \neq 0$, there exists a real number $\frac{1}{a}$ (read the *reciprocal* or *inverse* of $a$, or *one over* $a$), such that

$$(\tfrac{1}{a})a = a(\tfrac{1}{a}) = 1.$$

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[1] The algebra axioms are precisely the axioms for a *field*.

We begin to derive consequences of the axioms.

(2.1) (Cancellation law for addition.) If $a + b = a + c$ then $b = c$.

Proof: By the substitution principle (hereinafter abbreviated SP) we obtain

$$(-a) + (a + b) = (-a) + (a + c).$$

Apply (F.2) to both sides to get

$$\big((-a) + a\big) + b = \big((-a) + a\big) + c.$$

Applying (F.6) to both sides we obtain

$$0 + b = 0 + c,$$

and by (F.4) we have

$$b = c$$

as required.

(2.2) If $a + b = 0$ then $b = -a$.

Proof: By (F.6) we have $a + (-a) = 0$. By (SP) we have

$$a + b = a + (-a)$$

and by (2.1) we have

$$b = -a.$$

(2.3)* $-(-a) = a$.

(2.4) The equation $a + x = b$ has the unique solution $(-a) + b$.

Proof: First we verify that $(-a) + b$ is a solution, that is, if we calculate

$$a + [(-a) + b] = [a + (-a)] + b \qquad \text{by (F.2)},$$

$$[a + (-a)] + b = 0 + b \qquad \text{by (F.6)},$$

$$0 + b = b \qquad \text{by (F.4)},$$

we have checked that

$$a + [(-a) + b] = b.$$

To prove that the solution is unique, we suppose $x$ and $x'$ are solutions of the equation. Then we have

$$a + x = b$$

and

$$a + x' = b.$$

By (SP) we obtain

$$a + x = a + x',$$

and by the Cancellation Law,

$$x = x'.$$

Definition. The unique solution of the equation $x + a = b$ is denoted by $b - a$, and is called the operation of subtraction of $a$ from $b$, or $b$ minus $a$.

Thus, $(b - a) + a = b$.

(2.5)* $a - (b + c) = (a - b) - c$.

(2.6)* $-(a + b) = (-a) + (-b)$.

(2.7) $a \cdot 0 = 0$ for all real numbers $a$.

Proof: We have $1 + 0 = 1$ by (F.4). By (F.3) and (SP) we have

$$a(1 + 0) = a\cdot 1 + a\cdot 0 = a\cdot 1 = a\cdot 1 + 0.$$

By the Cancellation Law (2.1) we have

$$a\cdot 0 = 0.$$

(2.8) $(-a)b = -(ab)$, and in particular $(-1)b = -b$.

Proof: By (F.6) we have $a + (-a) = 0$. By (SP), (F.3), and (2.7) we have

$$\big(a + (-a)\big)b = ab + (-a)b = 0\cdot b = 0.$$

By (2.2) we have $(-a)b = -(ab)$ as required.

(2.9) $(-a)(-b) = ab$.

Proof: By two applications of (2.8) we have

$$(-a)(-b) = -\big(a(-b)\big) = -\big(-(ab)\big);$$

and by (2.3), $-\big(-(ab)\big) = ab$, as we wished to prove.

(2.10)* $(-1)(-1) = 1$.

Now we come to consequences of the "division axiom" (F.7). The starred theorems at the beginning are the exact parallels of (2.1) - (2.4) and it will be instructive for the reader to supply proofs.

(2.11)* (Cancellation Law for Multiplication.) If $ab = ac$ and $a \neq 0$, then $b = c$.

(2.12)* If $ab = 1$ then $b = \frac{1}{a}$.

(2.13)* $\frac{1}{\frac{1}{a}} = a$.

(2.14)* The equation $ax = b$, $a \neq 0$, has the unique solution $(\frac{1}{a})b$.

Definition. The unique solution of $ax = b$, $a \neq 0$, is called the result of dividing $b$ by $a$, and is denoted by $\frac{b}{a}$. We call $\frac{b}{a}$ a fraction with numerator $b$ and denominator $a$.

We can now settle the time honored question of division by 0. $\frac{a}{0}$ means the solution of the equation $0 \cdot x = a$. By (2.7), $0x = 0$ for all $x$, and the equation has no solution if $a \neq 0$. If $a = 0$, we have to consider the equation $0x = 0$, which is satisfied by every real number $x$. Therefore the equation $0x = a$ either has no solution (if $a \neq 0$) or infinitely many solutions (if $a = 0$). In neither case can we attach an unambiguous significance to $\frac{a}{0}$ which is consistent with the preceding definition of $\frac{a}{b}$. From now on, when we write $\frac{a}{b}$, it is tacitly assumed that $b \neq 0$.

(2.15)* If $ab = 0$ then either $a = 0$ or $b = 0$.

(2.16)* $(\frac{a}{b})(\frac{c}{d}) = \frac{ac}{bd}$.

(2.17)* $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$.

(2.18)* $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$.

(2.19)* $\dfrac{\frac{a}{b}}{\frac{c}{d}} = \frac{ad}{bc}$.

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3. *Order Axioms and Inequalities*. *Order Axioms*. There exists a collection of real numbers, called the *positive real numbers* with the following properties:

(0.1) for each real number $a$, one and only one of the following possibilities holds:

(a) $a$ is positive,

(b) $a = 0$,

(c) $-a$ is positive;

(0.2) if $a$ and $b$ are positive, so are $a + b$ and $ab$.

*Definition*. We shall write $a > 0$ for the statement that $a$ is positive, $a \geq 0$ for the statement that $a$ is either positive or zero. For any pair of real numbers $a, b$ we write $a < b$ (and read $a$ *is less than* $b$) if and only if $b - a$ is positive. We write $a \leq b$ (read $a$ *is less than or equal to* $b$) if either $a < b$ or $a = b$, and $a > b$ for the statement $b < a$. When $a > b$, we read $a$ *is greater than* $b$.

(3.1) For any two real numbers $a$ and $b$, one and only one of the following statements holds:

(a) $a < b$.

(b) $a = b$.

(c) $a > b$.

Proof: By (0.1) we have one and only one of the following possibilities:

$$b - a > 0$$

$$b - a = 0$$

$$-(b - a) = a - b > 0,$$

and these conditions are equivalent to (a), (b), and (c), respectively.

(3.2) If $a < b$ then $a + c < b + c$ for all real numbers $c$.

Proof: We have $(b + c) - (a + c) = b - a > 0$.

(3.3) If $a < b$ and $c > 0$ then $ac < bc$.

Proof: We have

$$bc - ac = (b - a)c > 0$$

by (0.2) and the fact that both $b - a$ and $c$ are positive by assumption.

(3.4) If $a < b$ and $c < 0$ then $ac > bc$.

Proof: We prove first that if $a < b$, then $-a > -b$. In fact, $-a - (-b) = b - a > 0$ since $a < b$. To prove (3.4), we obtain from $c < 0$ and what has just been proved, $-c > -0 = 0$. By (3.3), $a < b$ and $-c > 0$ imply $(-c)a < (-c)b$. By (2.8) this is equivalent to $-(ca) < -(cb)$. By the remark at the beginning of the proof, we have $-(-ca) > -(-cb)$, and by (2.3), $ca > cb$. This completes the proof of (3.4).

(3.5) If $a < b$ and $b < c$ then $a < c$. (Transitive Law.)

Proof: We have by hypothesis

$$b - a > 0$$

and

$$c - b > 0.$$

By (0.2) their sum is positive, or in other words

$$(b - a) + (c - b) = c - a > 0.$$

Thus, $a < c$.

(3.6) If $a \neq 0$ then $a^2 > 0$.

Proof: By (0.1) either $a > 0$ or $-a > 0$. In the first case, $a^2 > 0$ by (0.2). In the second, $(-a)^2 > 0$ by (0.2) and we have by (2.8) that $(-a)^2 = a^2 > 0$.

(3.7) $1 > 0$.

Proof: $1^2 = 1 \neq 0$, and apply (3.6).

(3.8)* $\frac{a}{b} > 0$ if and only if $ab > 0$.

We give now a definition which will play an important role in our development of geometry.

Definition. The notation

$$x < y < z$$

means that both inequalities $x < y$ and $y < z$ hold simultaneously. A real number $y$ is said to be between the real numbers $x$ and $z$ if either $x < y < z$ or $z < y < x$.

Now we come to the problem of solution of inequalities. We shall treat this problem by a number of examples.

Example 1. Find all real numbers $x$ such that $4x - 5 \geq 2x + 7$. We proceed by analogy with the solution of the linear equation $4x - 5 = 2x + 7$, but in the case of the inequality, we base our approach on (3.2) - (3.4), paying particular attention to the fact that when we multiply an equality by a negative number, we reverse the sense of the inequality. Thus, we obtain first by adding $5$ to both sides according to (3.2) that

$$4x \geq 2x + 12.$$

By (3.3) again we obtain

$$2x \geq 12.$$

Since $1$ and $2 = 1 + 1$ are both positive by (3.7), $\frac{1}{2} > 0$ by (3.8). Therefore by (3.3) we have

$$\tfrac{1}{2}(2x) \geq \tfrac{1}{2}(12)$$

and $x \geq 6$. Working backwards we can see that all numbers $x \geq 6$ do satisfy the original inequality, so that the solution of our problem is that those real numbers $x$ such that $4x - 5 \geq 2x + 7$ are precisely the real numbers $x$ such that $x \geq 6$.

Example 2. Solve the inequality

$$2x + 1 > 5x + 3.$$

As in Example 1, we may add $-5x$ to both sides and then $-1$ to both sides obtaining first

$$-3x + 1 > 3$$

and then

$$-3x > 2.$$

By (3.8) we know that $\frac{1}{-3} < 0$, and by (3.4) we have

$$(\frac{1}{-3})(-3x) < (\frac{1}{-3})2.$$

Simplifying we have

$$x < -\frac{2}{3},$$

and the set of all such $x$ constitutes the solution of our problem.

Example 3. Solve

$$\frac{1}{x - 1} > 0.$$

Solution: Our first temptation is to multiply by $x - 1$, and we are led to the confusing result $1 > 0$. What is wrong with our procedure? The point is that the inequality between

$$(x - 1)(\frac{1}{x - 1})$$

and $(x - 1)0$ is left in doubt because we do not know in advance whether $x - 1 > 0$ or $x - 1 < 0$. Thus, we must be more careful. First of all $x = 1$ cannot be a solution. By (3.1) we must have either

(a) $x - 1 > 0$, in which case we obtain $1 > 0$, or

(b) $x - 1 < 0$, in which case we have

$$1 < 0.$$

The second of these possibilities is ruled out by (3.7), and we are forced to conclude that $x - 1 > 0$ is the solution of our problem.

## Exercises

1. Solve the inequalities:

(a) $2x + 3 \geq 1$

(b) $3 - 2x \leq 4$

(c) $\frac{1}{x + 1} < 2$

(d) $\frac{1}{2x + 3} < 4$

(e) $1 < \frac{1}{x + 1} < 3$

(f) $\frac{x + 1}{4x - 3} < 1$

(g) $(2 - 3x)(3 + 2x) < 0$

(h) $(x - 2)(3 + x) > 0$

2. Prove the following statements:

(a) If $0 < a < b$, then $a^2 < b^2$.

(b) If $a$ and $b$ are both positive and $a^2 < b^2$, then $a < b$. What conclusion can you draw from the statement $a^2 < b^2$ by itself.

(c) If $a < b$, then $a < \frac{a + b}{2} < b$.

(Most of these are taken from Begle, Introductory Calculus, New York, 1954.)

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4. Absolute Value. Next we come to the important notion of absolute value. In the introductory discussion we saw that if on the edge of a ruler we read $a$ at one point and $b$ at another point, then the distance between these points should be $a - b$ or $b - a$, whichever is positive. It is worthwhile studying this situation in the light of the following definition.

Definition. For each real number $a$, we define

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0. \end{cases}$$

The number $|a|$ is called the absolute value of $a$.

Note that in the example above, the distance between the points on the ruler at which we read $a$ and $b$ is in both cases equal to $|a - b|$. We proceed to derive some properties of absolute value.

(4.1) For any real number $a \neq 0$, $|a| > 0$; $|a| = 0$ if and only if $a = 0$.

Proof: As in most proofs on absolute value we distinguish two cases.

(a) $a > 0$; then $|a| = a > 0$

(b) $a < 0$; then $-a > 0$ and $|a| = -a > 0$.

The second assertion is clear from the definition.

(4.2)* $|ab| = |a||b|$, $|-a| = |a|$.

The proof is by distinguishing cases.

(4.3)* $-|a| \leq a \leq |a|$.

A useful property of absolute value is contained in the following result:

(4.4) Let $b > 0$. Then $|a| < b$ if and only if

$$-b < a < b.$$

Proof: We recall that $-b < a < b$ means that both inequalities $-b < a$ and $a < b$ hold. Suppose first that $-b < a < b$. Then $|a| = a$ or $-a$, and in the first case we have $|a| < b$ because $a < b$. If $|a| = -a$, then we have $|a| = -a < b$ since $-b < a$.

Conversely, suppose that $|a| < b$. If $a \geq 0$, then $-b < 0$, and $-b < a$ by the transitive law (3.5), while $|a| = a < b$ by assumption. If $a < 0$, then $|a| = -a$, and $|a| < b$ implies $-a < b$ or $a > -b$. Finally, $a < 0$ and $0 < b$ imply $a < b$, again by (3.5). This completes the proof.

(4.5) $|a + b| \leq |a| + |b|$.

Proof: By (4.3) we have

$$-|a| \leq a \leq |a|$$
$$-|b| \leq b \leq |b|.$$

Then

$$-(|a| + |b|) \leq a + b \leq |a| + |b|$$

and we have by (4.4) $|a + b| \leq |a| + |b|$ as required.

It will be important for us to solve equations and inequalities involving absolute value.

Example 1. Solve

$$|x - 2| = |4 - x|.$$

We have $|x - 2| = \pm (x - 2)$, while $|4 - x| = \pm (4 - x)$. From $|x - 2| = |4 - x|$ we have four apparently different equations:

$$\begin{aligned} x - 2 &= 4 - x \\ -(x - 2) &= 4 - x \\ x - 2 &= -(4 - x) \\ -(x - 2) &= -(4 - x). \end{aligned}$$

These reduce to the two equations

$$x - 2 = 4 - x \quad \text{and} \quad x - 2 = -(4 - x) = x - 4.$$

The first has the solution $x = 3$, while the second has no solution. Checking in the original equation we see that $x = 3$ is the unique solution of the equation.

Example 2. Solve

$$|x - 4| < 3.$$

Solution: By (4.4), $|x - 4| < 3$ if and only if

$$-3 < x - 4 < 3.$$

Thus, $x - 4$ must simultaneously satisfy the inequalities

$$x - 4 > -3$$

and

$$x - 4 < 3.$$

The solution of the first is $x > 1$ and the solution of the second is $x < 7$. The solution of the original inequality $|x - 4| < 3$ is therefore the collection of all real numbers $x$, such that

$$1 < x < 7.$$

Example 3. What conclusions can be drawn from the equality $|a - b| = |b - c|$ if it is known that $a \neq c$?

Solution: The equality can mean one of four things,

(i) $a - b = b - c$,

(ii) $a - b = c - b$,

(iii) $b - a = b - c$,

(iv) $b - a = c - b$.

Notice that (i) and (iv) say the same thing, namely,

$$a + c = 2b;$$

and so do (ii) and (iii), namely,

$$a = c.$$

Since we are given that $a \neq c$ the conclusion is that $a + c = 2b$.

## Exercises

1. Solve the equations or inequalities:

(a) $|x + 2| < 3$

(b) $|2x - 1| < 1$

(c) $|2 - x| = 1$

(d) $|4x + 1| = 7$

(e) $|2x + 3| = |4 - x|$.

(f) $|2 - x| > \frac{1}{2}$

(g) $|x + 1| < a$

(h) $|x - c| < a$

(These exercises are taken from Begle, Introductory Calculus, New York, 1954.)

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5. Completeness Axiom. Let us first explain some familiar notions in the context of the real number system. By (3.7), 1 is a positive number. By (0.2), the numbers

$$2 = 1 + 1, \quad 3 = 2 + 1, \quad 4, \quad 5, \quad \ldots$$

are all positive. These numbers are called the *natural numbers*, and as we have seen they are all positive. If $a$ and $b$ are natural numbers, so are $a + b$ and $ab$, but not necessarily $a - b$ or $\frac{a}{b}$.

Next, we define the collection of all *integers* to be the natural numbers and their negatives, together with zero. If $a$ and $b$ are integers, then so are $a + b$, $ab$, and $a - b$, but not in general $\frac{a}{b}$.

We extend the system of integers still further to the *system of rational numbers*, where the rational numbers are those real numbers which can be expressed in the form $\frac{a}{b}$ where $a$ and $b$ are integers and $b \neq 0$. We can prove by (2.16), (2.17), and (2.19) that if $r$ and $s$ are rational numbers so are $r \pm s$, $r \cdot s$, and $\frac{r}{s}$ (if $s \neq 0$).

Two remarks are in order. First, that the system of rational numbers satisfies all the axioms we have had up to now. On the other hand, we saw in the introduction to this chapter that if we tried to measure the lengths of all line segments with rational numbers, we would be unable to attach a length to the hypotenuse of an isosceles right triangle whose other sides have unit length. To make this remark precise, we prove the following result.

(5.1) There is no rational number $x$, such that $x^2 = 2$.

Proof: This argument is a famous example of the method of indirect proof. We suppose the result is false, and, hence, that there does exist a rational number $\frac{a}{b}$ where a and b are integers, such that

$$(\frac{a}{b})^2 = 2.$$

We may assume that a and b have no common factors besides $\pm 1$, for any common factor can be divided out without changing $\frac{a}{b}$. Expanding the first equation yields

$$a^2 = 2b^2,$$

and consequently $a^2$ is an even number. a itself is either odd or even, and if odd we can express $a = 2m + 1$ for some integer m. Then $a^2 = 4m^2 + 4m + 1 = 2(2m^2 + 2m) + 1$, so that the square of an odd number is odd. Because $a^2$ is even we conclude that a is even and can write $a = 2c$ for some integer c. Substituting in the last equation we have

$$4c^2 = 2b^2,$$

and

$$2c^2 = b^2.$$

Thus, $b^2$ is even, and by the same argument used for a, we know that b is even. We have shown that a and b have the common factor two, even though we had previously guaranteed that a and b had no common factors other than $\pm 1$. The only conclusion left is that our original assumption

$$(\frac{a}{b})^2 = 2$$

is false, in other words the equation $x^2 = 2$ has no solution in the system of rational numbers.

We give now an axiom which guarantees that the real number system does contain enough numbers to serve as the basis for measurement in geometry.

Definition. A collection of real numbers $S$ is bounded above if there exists a real number $M$, such that $s \leq M$ for every $s$ in $S$. The number $M$ is called an upper bound of $S$. A number $L$ is a least upper bound of the set $S$ if (a) $L$ is an upper bound of $S$ and (b) if $M$ is any upper bound of $S$, then $L \leq M$.

Completeness Axiom. Every non-empty set of real numbers which is bounded above has a least upper bound.

(5.2)* If $L$ and $L'$ are least upper bounds of the set of real numbers $S$, then $L = L'$.

(5.3) There exists a real number $x_o$, such that $x_o^2 = 2$.

Proof: Let $S$ be the set of all positive real numbers $x$, such that $x^2 < 2$. The number 2, for example, is an upper bound for the set $S$, so that by the Completeness Axiom, $S$ has a least upper bound $x_o \leq 2$. Because $1^2 < 2$, we have $x_o \geq 1$. By (0.1), we must have either $x_o^2 - 2 > 0$, $x_o^2 - 2 < 0$, or $x_o^2 - 2 = 0$. We shall show that neither of the first two cases can occur.

First, suppose $x_0^2 < 2$, and let $h = 2 - x_0^2$. Then $0 < h \leq 1$. Let $x' = x_0 + \frac{1}{8}h$; then $x' > x_0$, and we shall prove that $(x')^2 < 2$. We have

$$\begin{aligned}
(x')^2 = (x_0 + \tfrac{1}{8}h)^2 &= x_0^2 + \frac{x_0 h}{4} + \frac{h^2}{64} \\
&= x_0^2 + \frac{h}{4}(x_0 + \frac{h}{16}) = x_0^2 + \frac{h}{4}(\frac{16x_0 + h}{16}) \\
&\leq x_0^2 + \frac{h}{4}(\frac{33}{16}) \qquad \text{(since } x_0 \leq 2 \text{ and } h \leq 1) \\
&< x_0^2 + h = 2 \qquad \text{(since } \frac{33}{64} < 1).
\end{aligned}$$

We have shown that $x'$ is a number in the set $S$ which is larger than $x_0$, contrary to our assumption that $x_0$ is an upper bound of $S$, and the possibility $x_0^2 < 2$ has been eliminated.

Next, suppose that $x_0^2 > 2$, and let $k$ be the positive number $x_0^2 - 2$. Then for any real number $u > x_0$,

$$\begin{aligned}
(x_0 - \frac{k}{2u})^2 - 2 &= x_0^2 - \frac{x_0(x_0^2 - 2)}{u} + (\frac{k}{2u})^2 - 2 \\
&= \frac{(u - x_0)(x_0^2 - 2)}{u} + (\frac{k}{2u})^2 > 0.
\end{aligned}$$

At the same time, if $u > \frac{k}{2x_0}$ then we have $x_0 > \frac{k}{2u}$ or $x_0 - \frac{k}{2u} > 0$. Now choose a real number $u$ which is greater than both $x_0$ and $\frac{k}{2x_0}$ and set

$$x_0' = x_0 - \frac{k}{2u}.$$

Then as we have seen,

$$0 < x_0' < x_0$$

and

$$(x_0')^2 > 2.$$

Now, let $x$ be any positive real number, such that $x^2 < 2$. Then

$$(x_0')^2 - x^2 > 0,$$

and, hence,

$$(x_0' - x)(x_0' + x) > 0.$$

Because $x_0'$ and $x$ are positive, $x_0' + x > 0$ by (0.2), and it follows that

$$x'_0 - x > 0$$

or $x'_0 > x$. Therefore $x'_0$ is an upper bound of the set $S$ which is actually less than $x_0$, contrary to our assumption that $x_0$ is the least upper bound of $S$. Therefore the possibility $x_0^2 > 2$ has also been ruled out, and we conclude finally that $x_0^2 = 2$. This completes the proof.

By the same method the following result can be proved.

(5.4) Let $a$ be a real number $> 0$. Then the equation $x^2 = a$ has exactly two solutions $\pm r$, where $r$ is a real number such that $r^2 = a$.

Definition. Let $a > 0$. Then the unique positive solution of the equation $x^2 = a$ will be called the square root of $a$ and denoted by $\sqrt{a}$; the other solution of the equation is therefore $-\sqrt{a}$.

As a consequence of these definitions we have

(5.5)* $\sqrt{a^2} = |a|$.

## Exercises

1. Prove that there exists a real number $x$, such that $x^2 = 5$.

2. The equation $ax^2 + bx + c = 0$ with real number coefficients $a, b, c$, $a \neq 0$, has a real number solution $x$ if and only if $b^2 - 4ac \geq 0$. There are one or two distinct solutions accordingly as $b^2 - 4ac = 0$ or $b^2 - 4ac > 0$.

3. Prove that $\sqrt{3}$ is an irrational number.

4. Is $\sqrt{3} + \sqrt{2}$ rational or irrational?

---

6. One-to-One Correspondences. As a last preparation for the ruler postulate we discuss the important concept of a one-to-one correspondence between sets, which has already been introduced from a more general point of view in Chapter 2. This notion originates in the problem of deciding when two sets of objects have the same number of elements. For example, at a dance, let $M$ be the set of all men present and $W$ the set of all women present. In order to decide whether the number of men in the set $M$ is equal to the number of women in the set $W$, we could count both numbers separately, and compare the results. Another method would be to wait until the dance began, and see whether every man and woman has a partner. It is the latter idea that leads to the concept of a one-to-one correspondence.

**Definition.** A one-to-one correspondence between two sets A and A' is a rule which assigns to every object a in A exactly one object a' in A' in such a way, that if $a_1 \neq a_2$ in A, then $a_1' \neq a_2'$ in A', and such that every object in A' has an element in A assigned to it by the rule.

We may think of the rule as a pairing of the objects in A with the objects in A', in such a way, that (a) each object in A is assigned exactly one partner from A'; (b) two different objects in A have different partners in A'; and (c), every object in A' is the partner of some object in A.

We shall often denote a one-to-one correspondence by the notation

$$a \longleftrightarrow a'$$

which means that a' is the partner of a assigned by the rule.

## Exercises

1. Let A be the set of all integers, and consider the rule that assigns to each integer its cube: $a \longleftrightarrow a^3$. Does this rule define a one-to-one correspondence of A with itself?

2. Let A be the set of all integers and consider the rule $a \longleftrightarrow a + 5$. Is this a one-to-one correspondence of A with itself?

3. Let A be the set of all integers and consider the rule $a \longleftrightarrow 2a$. Is this a one-to-one correspondence of A with itself?

4. Let R be the set of all real numbers, and consider the rule $a \longleftrightarrow 2a$. Is thus a one-to-one correspondence of R with itself?

---

7. The Ruler Postulate. After a long digression we are back to geometry again! This section contains the Ruler Postulate, which is actually broken down into three separate parts which we call Postulates 6, 7 and 8, some definitions, and the consequences of the Ruler Postulate for the study of the notion of betweenness for points on a line.

At this point the reader is advised to reread the introduction to this chapter, especially the motivating discussion for the Ruler Postulate.

Postulate 6. To every pair of points A, B there corresponds a unique real number, designated by AB, and called the distance between A and B. If A and B are different points then AB is positive. We allow also the possibility that $A = B$; in this case, $AB = 0$.

Postulate 7. The points of a line can be put in one-to-one correspondence with the real numbers in such a way, that the distance between two points is the absolute value of the difference betwaen the corresponding numbers.

(7.1) The distance AB has the properties that

$$AB = BA,$$

and if A, B, C are collinear, then

$$AC + CB = AB.$$

Proof: The first result follows from the fact that $|y - x| = |x - y|$. For the second, let A, B, C correspond to the real numbers a, b, c, respectively. Then

$$\begin{aligned} AC + CB &= |c - a| + |b - c| \geq |(c - a) + (b - c)| \\ &= |b - a| = AB \end{aligned}$$

by (4.5). This completes the proof.

Definition. A correspondence of the kind described in Postulate 7 is called a coordinate system on $\mathcal{L}$; and the number corresponding to a point of $\mathcal{L}$ is called the coordinate of that point.

For the convenience of the reader we repeat the following definition, already given in [illegible]

Definition. Let x, y, z be real numbers. Then z is said to be between x and y if either $x < z < y$ or $y < z < x$.

Definition. If A, B, C are different points on a line $\mathcal{L}$, B is said to be between A and C if and only if

$$AB + BC = AC.$$

Theorem 4.1. If the three different collinear points A, B, C have coordinates $x, y, z$, respectively, then B is between A and C if and only if $y$ is between $x$ and $z$.

Proof: If $y$ is between $x$ and $z$ then either $x < y < z$ or $z < y < x$. In the first case,

$$\begin{aligned} AB + BC &= |x - y| + |y - z| \\ &= (y - x) + (z - y) \\ &= z - x = |x - z| = AC, \end{aligned}$$

and B is between A and C. The case $z < y < x$ proceeds similarly.

Conversely, suppose B is between A and C, so that

$$AB + BC = AC.$$

Then

$$|x - y| + |y - z| = |x - z|.$$

There are eight possible cases of this equation:

$$\begin{aligned} (x - y) + (y - z) &= x - z \\ (x - y) + (y - z) &= z - x \\ (x - y) + (z - y) &= x - z \\ (x - y) + (z - y) &= z - x \\ (y - x) + (y - z) &= x - z \\ (y - x) + (y - z) &= z - x \\ (y - x) + (z - y) &= x - z \\ (y - x) + (z - y) &= z - x. \end{aligned}$$

The six middle cases lead to $x = y$, $x = z$, or $y = z$, none of which is true because A, B, and C are distinct points, and the correspondence between points and their coordinates is one-to-one. The first case arises when $x > y > z$ and the last when $x < y < z$, so that in both possible cases, $y$ is between $x$ and $z$.

*Theorem 4.2*$^*$. Of three different collinear points, precisely one is between the other two.

We remark that it may be possible to introduce many different coordinate systems on a line; intuitively, this means that we shift our ruler along the line in some way, or that we use a different ruler (i.e., we change the unit of length). We emphasize, however, that the definition of *betweenness* is independent of the choice of a particular coordinate system, and that the conclusion of Theorem 4.2 is independent of the choice of a coordinate system, although the proof is not.

In accordance with our remarks on page 71 we do not wish to change the unit of length, but we may wish to change the position of the ruler on the line. This is accomplished by the following postulate.

*Postulate 8*. If A and B are distinct points on a line $\ell$ then a coordinate system can be chosen on $\ell$, such that the coordinate of A is zero and the coordinate of B is positive.

8. *Segments and Rays*. We begin with some more definitions.

*Definition*. If A and B are different points on a line $\ell$, the union of A, B, and all points of $\ell$ between A and B is called the *segment* $\overline{AB}$. A and B are *end-points* of $\overline{AB}$. d(A,B) is the *length* of $\overline{AB}$.

*Definition*. If A and B are different points on a line $\ell$, the union of $\overline{AB}$ and all points P on $\ell$, such that B is between A and P is called the *ray* $\overrightarrow{AB}$. A is the *end-point* of $\overrightarrow{AB}$.

*Definition*. If A, B, and C are different points on a line $\ell$, and C is between A and B, the rays $\overrightarrow{CA}$ and $\overrightarrow{CB}$ are said to be *opposite*.

*Theorem 4.3*. Given a coordinate system on a line, the set of points whose coordinates are positive or zero is a ray, and the set of points whose coordinates are negative or zero is the opposite ray.

Proof: Let A have coordinate zero, B a positive coordinate x, C a negative coordinate y. If P is a point whose coordinate z is positive or zero, then by (0.1) either

$$
\begin{aligned}
z &= 0 \text{ and } P = A;\\
z &= x \text{ and } P = B;\\
z - x &> 0 \text{ and } B \text{ is between } A \text{ and } P; \text{ or}\\
-(z - x) &> 0 \text{ and } P \text{ is between } A \text{ and } B.
\end{aligned}
$$

In any case $P$ is a point of $\overrightarrow{AB}$. Conversely, if $P$ is a point of $\overrightarrow{AB}$ the four cases can be reversed to prove that $z$ is either positive or zero.

The relation between negative values of $z$ and the ray $\overrightarrow{AC}$ is proved in a similar way. Since $A$ is between $B$ and $C$, $\overrightarrow{AC}$ is opposite to $\overrightarrow{AB}$.

Theorem 4.4$^*$. The union of two opposite rays is the line containing them.

Theorem 4.5$^*$. A ray is determined by its end-point and any of its other points, i.e., if $C$ is a point of $\overrightarrow{AB}$ other than $A$, then $\overrightarrow{AC} = \overrightarrow{AB}$.

Theorem 4.6$^*$. Given a positive real number $r$, on any ray there is exactly one point whose distance from the ray's end-point is $r$.

Theorem 4.7$^*$. Given a line $\ell$, a point $P$ on $\ell$, and a positive real number $r$, there are exactly two points on $\ell$ whose distance from $P$ is $r$.

Theorem 4.8$^*$. If $P$ and $Q$ are points on the ray $\overrightarrow{AX}$, then $P$ is between $A$ and $Q$ if and only if $AP < AQ$.

Theorem 4.9$^*$. If $C$ and $D$ are different points of $\overline{AB}$ (or of $\overrightarrow{AB}$), then every point of $\overline{CD}$ is a point $\overline{AB}$ (or of $\overrightarrow{AB}$).

*Theorem 4.10*. If A and B are distinct points there is exactly one point M on line $\overleftrightarrow{AB}$, such that d(A,M) = d(B,M).

Proof: Let p and q be the coordinates of A and B in a coordinate system on $\ell$, and let x be the coordinate of M, assuming the M exists. If AM = BM we must have

$$|p - x| = |z - x|.$$

There are two cases (the reader will recall a discussion of this problem in Example 3 of §4).

(i) $p - x = q - x$.

This gives $p = q$, which is impossible since A and B are different points.

(ii) $p - x = x - q$.

This equation has the unique solution $x = \frac{1}{2}(p + q)$, and the unique point M on $\ell$ with this coordinate is the point required in the statement of the theorem.

*Definition*. Let A and B be different points on a line $\ell$. The unique point M on $\ell$ with the property that AM = BM is called the *mid-point* of the segment $\overline{AB}$.

*Definitions*. If A, B, C are non-collinear points, the union of $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ is called a *triangle*. These three segments are called the *sides* of the triangle; the points A, B, C the *vertices* of the triangle. A vertex and a side are said to be *adjacent* if the side contains the vertex, otherwise the vertex and the side are said to be *opposite*. We shall denote the triangle with vertices A, B, C by the notation $\Delta$ ABC.

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## Chapter 5

## Separation in Planes and in Space

1. Introductory Remarks. In Sections 7 and 8 of the preceding chapter, the order of points on a line was discussed with the help of a precise definition of the relation of betweenness for points on a line. One by-product of our efforts is that the reader now has the tools to formulate meaningful definitions and problems concerning the order of points on a line, besides those that have already been introduced. For example, the reader is invited to give a precise meaning to the statement concerning the three points A, B, C on a line, that A and B are on the same side of C.

Corresponding problems can now be raised concerning lines in the plane or in space. For example, given three distinct coplanar rays $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$, the reader will observe that at this point he is unable to give meaning to the statement that the ray $\overrightarrow{OB}$ is between the rays $\overrightarrow{OA}$ and $\overrightarrow{OC}$. Or given a line $\ell$ in a plane p, what does it mean to say that two points A and B in the plane p both lie on the same side of $\ell$? Other questions which will present themselves later are these: What is meant by the interior of an angle? What is the interior of a triangle?

Most of the important theorems on angles and triangles cannot be established in a satisfactory way without coming to grips with these "separation" problems in the plane. This chapter contains only the basic separation axioms and their consequences, but the subject will demonstrate its importance and power time and time again in Chapter 6 on angles and Chapter 7 on congruence.

The reader may be concerned about our insistence upon a detailed study of seemingly trivial points. To this we say first that the reader must admit that the difficulties we consider do exist, so that it is only natural that we should confront them forthrightly. There is also the point that the geometry we are developing is intended to be a mathematical model of our perceptual geometry. Thus some of the difficulties we face are concerned with whether our mathematical model is appropriate, and the fact that we can overcome them gives us confidence that our mathematical geometry is in agreement with our intuition.

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2. *The Separation Postulates in the Plane and in Space*. There are two separation postulates, the first describing the separation of a plane into two half-planes by a line; and the second, the separation of space into two half-spaces by a plane.

*Postulate 9*. If $\ell$ is a line and $p$ a plane containing $\ell$, the points of $p$ not in $\ell$ consist of two non-empty sets, called *half-planes*, such that if two points $X$ and $Y$ are in the same half-plane the segment $\overline{XY}$ does not intersect the line $\ell$; and if $X$ and $Y$ are in different half-planes, the segment $\overline{XY}$ does intersect the line $\ell$

Postulate 10. If p is a plane, the points not in p consist of two non-empty sets, called half-spaces, such that if two points X and Y are in the same half-space, the segment $\overline{XY}$ does not intersect the plane p; and if X and Y are in different half-spaces, the segment $\overline{XY}$ does intersect the plane.

Postulate 9 can be proved from Postulate 10, but it is in keeping with our elementary approach to present these separation properties as distinct postulates.

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## 3. Theorems on Separation in the Plane and in Space.

We begin with some definitions.

Definition. Two points in the same half-space determined by a plane p are said to be on the same side of p; two points in different half-spaces determined by p are said to be on opposite sides of p. Similarly, two points in a plane are on the same or opposite sides of a line in the plane according as they lie in the same or different half-planes determined by the line.

Definition. A line is said to be the edge of any of the half-planes determined by it. A plane is the face of either of the half-spaces determined by it.

First of all we observe that the "if" statements in Postulates 8 and 8a can be replaced by "if and only if." We have:

(3.1)* Two different points A and B lie in the same half-space or half-plane h if and only if $\overline{AB}$ does not intersect the face or edge, respectively, of h.

Theorem 5.1. If h is a half-plane or a half-space, and if A and B are two different points in h, then every point of $\overline{AB}$ is in h.

Proof: Assume first that h is a half-plane and let $\ell$ be the edge of h. Let C be a point of $\overline{AB}$; if $C = A$, then C is in h by assumption, so that we may assume $C \neq A$. By Theorem 4.9 every point of $\overline{AC}$ is a point of $\overline{AB}$. By (3.1) no point of $\overline{AB}$ is on $\ell$, and consequently no point of $\overline{AC}$ is on $\ell$. By (3.1) again, A and C are on the same side of $\ell$. Because A is in h, so is C, and the proof for the case of a half-plane is completed. The proof in the case of a half-space is entirely analogous to the argument we have given, and the details will be omitted.

Theorem 5.2*. If h is a half-plane with edge $\ell$ or a half-space with face $\ell$ and if A is a point $\ell$ and B a point of h, then every point of the ray $\overrightarrow{AB}$, other than A, is in h.

For convenience, when the conditions of Theorem 5.2 are satisfied we shall say that $\overrightarrow{AB}$ lies in h, although this is not strictly true because A is not in h.

Theorem 5.3. In a plane p consider six points with the following relationships among them. A, B, and C are non-collinear, B is between A and D, E is between B and C and also between A and F. Then in the plane p, C and F are on the same side of $\overleftrightarrow{AB}$ and D and F are on the same side of $\overleftrightarrow{BE}$.

[This theorem is used in Chapter 7 to prove that the measure of an exterior angle of a triangle is greater than the measure of either of the remote interior angles.]

Before beginning the proof the reader is advised to make a figure to fix the relationships in his mind. It is also helpful to derive the following preliminary result.

(3.2) If in a plane p, C and E lie on the same side of a line $\ell$ in p, and if E and F lie on the same side of $\ell$, then C and F lie on the same side of $\ell$.

Proof of (3.2): Let h be the half-plane with edge $\ell$ containing C and E. Then C and F also lie in h. Therefore, E and F lie on the same side of $\ell$.

Proof of Theorem 5.3: From the hypothesis of the theorem, the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$ are distinct and intersect in the unique point B. On $\overleftrightarrow{BC}$, E is between B and C. By Theorem 4.2, B is not between C and E, Consequently no point of the segment $\overline{CE}$ lies on $\overleftrightarrow{AB}$ and it follows that C and E are on the same side of $\overleftrightarrow{AB}$. Because E is not on $\overleftrightarrow{AB}$, $\overleftrightarrow{AF}$ and $\overleftrightarrow{AB}$ are distinct lines intersecting at A. Because E is between A and F, A is not between E and F and we conclude as before that E and F are on the same side of $\overleftrightarrow{AB}$. By (3.2), C and F are on the same side of $\overleftrightarrow{AB}$, and the first assertion is proved.

For the second part, observe that the segments $\overline{AF}$ and $\overline{AD}$ both intersect the line $\overleftrightarrow{BE}$ in the points B and E respectively.

Moreover none of the points A, D, or F lie on the line $\overleftrightarrow{BE}$. Therefore A and F lie on opposite sides of $\overleftrightarrow{BE}$ and, A and D lie on opposite sides of $\overleftrightarrow{BE}$. Let h and h' be the half-planes determined by the line $\overleftrightarrow{BE}$. If A is in h, then both F and D lie in h' and we conclude that F and D lie on the same side of the line $\overleftrightarrow{BE}$. This completes the proof of this Theorem.

Theorem 5.4. (Axiom of Pasch). If a line in the plane of $\Delta ABC$ intersects $\overline{AB}$ in a point D between A and B, then $\ell$ either contains C or A, or $\ell$ intersects $\overline{AC}$ but not $\overline{BC}$, or $\ell$ intersects $\overline{BC}$ but not $\overline{AC}$.

Proof: If $\ell$ contains C or A then there is nothing to prove. Thus we may assume that $\ell$ does not contain C, and intersects $\overline{AB}$ in the unique point D. By (3.1), A and B are on opposite sides of $\ell$. Let h and h' be the half-planes in the plane of the triangle ABC with edge $\ell$, and suppose that A is in h, B in h'. Because C is not on $\ell$, C is either in h or in h'. If C is in h', then A and C are on opposite sides of $\ell$, and $\ell$ intersects $\overline{AC}$, while because B and C are on the same side of $\ell$, $\overline{BC}$ does not intersect $\ell$. Similarly, if C is in h, then $\ell$ intersects $\overline{BC}$ but not $\overline{AC}$. This completes the proof.

In his approach to geometry, Hilbert used Theorem 5.4 as a postulate in place of our Postulate 9.

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4. *Convex Sets in the Plane*. In a way this section is an appendix to the chapter; it introduces the reader to some ideas which have proved to be of great importance in present day mathematics, but which lead very rapidly outside the realm of elementary geometry. On the other hand, it is possible to develop at least the simplest parts of the subject as an application of the work we have done in this chapter.

Throughout the section, we shall consider sets of points lying in a fixed plane p.

*Definition*. A set of points C in the plane p is said to be *convex* if whenever A and B belong to C, so does the entire segment $\overline{AB}$.

The simplest example of a convex set is the whole plane p. A less trivial example is given by the following remark.

(4.1) A half-plane is convex.

This assertion is merely a restatement of Theorem 5.1. In fact the reader may verify that the first part of Postulate 9 could be replaced by the equivalent statement:

If $\ell$ is a line and p a plane containing $\ell$, the points of p not in $\ell$ consist of two non-empty convex sets, called *half-planes*.

One of the most useful properties of convex sets is the following:

(4.2)* The intersection of two or more convex sets is convex.

Question: Is the union of two convex sets always convex?

**Definition**. Let A, B, C be three non-collinear points. Let $H_A$ be the half-plane with edge $\overleftrightarrow{BC}$ containing A, $H_B$ the half-plane with edge $\overleftrightarrow{AC}$ containing B, and $H_C$ the half-plane with edge $\overleftrightarrow{AB}$ containing C. The intersection of the half-planes $H_A$, $H_B$, $H_C$ is called the interior of the triangle ABC.

(4.3)* The interior of a triangle is a convex set.

## Exercises

1. Prove Postulate 9 from Postulate 10. (Hint: Let q be a plane distinct from p and containing $\ell$ , and apply Postulate 10 to q.)

2. Determine all convex sets which are contained in a line $\ell$ .

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# Chapter 6

## Angles and the Protractor Postulates.

1. Introduction. In Chapter 4 we were careful to distinguish between the concepts of a segment and its length; a segment was defined as a point set, and its length as the real number measuring the distance between the end-points according to the Ruler Postulate The approach we shall adopt in this chapter is similar to the discussion in Chapter 4. Thus an angle is defined as a geometrical object, i.e., a point set, consisting of the union of two non-collinear rays with the same end-point. With every angle is associated a unique real number which we shall call the measure of the angle. The properties of this measure are stated, for convenience in four postulates, which together constitute an abstract "protractor," just as Postulates 6, 7 and 8 specified a "ruler". That is, we assume that we are given once and for all a "protractor", a segment together with a semi-circle with the segment as diameter, such that at each point on the circumference of the semi-circle is marked a real number from 0 to 180, and such that each real number from 0 to 180 has its mark on our protractor. Then an angle consisting of the non-collinear rays $\overrightarrow{QA}$ and $\overrightarrow{QB}$ can be measured by laying down the protractor in such a way that Q lies at the mid-point of the straight edge of the protractor and A and B lie in the half-plane determined by the extended edge of the protractor and containing the marked semi-circle on the protractor. Then the rays $\overrightarrow{QA}$ and $\overrightarrow{QB}$ will intersect the arc

of the protractor in two points, at which we may read the real numbers a and b respectively. The measure of the angle AQB then is given by $|a - b|$.

We emphasize that as in our discussion of distance, we are adopting a fixed unit of angular measure, which we shall call the degree. Of course it is possible to replace 180 by any other positive real number and thus change the unit of angular measure. As the reader may verify, however, such a change will not affect the statements or content of our theorems in any significant way.

The one big difference between measure of distance and measure of angle lies in the fact that any positive number is the measure of some distance, whereas the measure of an angle is restricted to a limited range of numbers, 0 to 180 if we use the degree as the unit of measure. In general this makes angles more difficult to deal with than segments. One way to get around this restriction is to define "angle" differently, so that it is no longer merely a point set. This introduces other difficulties but it is found to be essential for a complete treatment of trigonometry.

It is worth knowing that in one of the so-called "non-Euclidean" geometries the measure of distance is exactly analogous to our measure of angle. Going in the other direction, one might try to develop a geometry in which both distance and angle are allowed to have arbitrarily large values. It turns out, however, that it is impossible to do this and still preserve the basic incidence postulates of Chapter 3.

2. Definitions and the Separation Properties. All the definitions and results lean heavily on the material on separation given in Chapter 5.

Definitions. An angle is the union of two non-collinear rays with the same end-point. The two rays are called the sides of the angle; their common end-point the vertex of the angle.

Notations. If $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are non-collinear rays with common end-point $B$, we shall denote the angle formed by them by $\angle ABC$ (read "angle $ABC$"). Conversely the notation $\angle ABC$ will be used whenever $A, B, C$ are non-collinear to denote the angle which is the union of the rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$. Sometimes we shall abbreviate $\angle ABC$ to $\angle B$ when there is no possibility of confusion.

We remark that if $A, B, C$ and $A', B', C'$, are two sets of non-collinear points, we have

$$\angle ABC = \angle A'B'C'$$

If and only if the union of the rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$ is the same point set as the union of the rays $\overrightarrow{B'A'}$ or $\overrightarrow{B'C'}$. This requires that $B = B'$, $A$ lies on $\overrightarrow{B'A'}$ or $\overrightarrow{B'C'}$, etc.

Definition. The interior of $\angle ABC$ is the intersection of the half-planes $h_A$ and $h_C$, where $h_A$ is the half-plane with edge $\overleftrightarrow{BC}$ containing $A$ and $h_B$ is the half-plane with edge $\overleftrightarrow{AB}$ containing $C$.

*Definition*. The *exterior* of an angle is the set of all points in the plane containing the angle which are neither on the angle nor in the interior of the angle.

*Definition*. Two angles are said to be *adjacent* if they both lie in the same plane and if they have a common side such that their other sides lie in opposite half-planes determined by the line containing their common side.

*Definition*. The *vertical angle* of the angle ABC is the angle formed by the rays opposite to $\overrightarrow{BA}$ and $\overrightarrow{BC}$.

*Definition*. Given a triangle ABC, the angles $\angle BAC$, $\angle ACB$, and $\angle CBA$ are called *the angles of the triangle*.

Note that an angle of a triangle is not a subset of the triangle. $\angle ABC$ is the union of the two rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$, whereas $\Delta ABC$ contains only portions of these rays, namely segments $\overline{BA}$ and $\overline{BC}$.

The following theorems state and prove the fundamental separation properties of angles. Most of these properties are "intuitively obvious" and at the same time rather difficult to prove. (This is a common situation in mathematics.) These proofs are therefore not appropriate for presenting to most high school students. In the text book these "obvious" properties would be assumed. However, a good teacher should realize that they require proof (unless we introduce them as additional postulates) and should be able to prove them.

Theorem 6.1. Any point between A and B is in the interior of $\angle AQB$.

Proof: Let P be a point between A and B. By Theorem 5.2, every point on the ray $\overrightarrow{BA}$ other than B lies in the half-plane with edge $\overleftrightarrow{QB}$ containing A, and since P lies on the ray $\overrightarrow{BA}$, P is in this half-plane. Similarly P lies in the half-plane with edge $\overleftrightarrow{QA}$ containing B, and so by the definition at the bottom of page 117, P is in the interior of $\angle AQB$.

Theorem 6.2. If $\overrightarrow{QB}$ and $\overrightarrow{QC}$ are different rays in a half-plane with edge $\overleftrightarrow{QA}$ then either B is in the interior of $\angle AQC$ or C is in the interior of $\angle AQB$.

Proof: First we sketch the basic idea of the proof. We have to prove that either one of two statements is valid. Consider the first statement: it is either true or false. If it is true, then there is nothing to be proved. If it is false, then we must prove that the second statement is true. Therefore it is sufficient to prove that if B is not in the interior of $\angle AQC$, then C is in the interior of $\angle AQB$.

Let h be the half-plane with edge $\overleftrightarrow{QA}$ containing B and C. Because the interior of $\angle AQC$ is the intersection of h with the half-plane h' with edge $\overleftrightarrow{QC}$ containing A, B does not belong to the interior of $\angle AQC$ only if B is not in h'. If this occurs we must prove that C belongs to the interior of $\angle AQB$.

Thus we suppose $B$ is not in $h'$; then $A$ and $B$ are on opposite sides of $\overleftrightarrow{QC}$, and $\overline{AB}$ intersects $\overleftrightarrow{QC}$ in a point $R \neq Q$.

Our first objective is to show that the rays $\overrightarrow{QR}$ and $\overrightarrow{QC}$ are identical. The only way this can be false is for $\overrightarrow{QR}$ and $\overrightarrow{QC}$ to be opposite rays. In this case, $Q$ is between $R$ and $C$, and since $C$ is in the half-plane $h$, $R$ is in the half-plane $h''$ with edge $\overleftrightarrow{QA}$ opposite to $h$. By Theorem 5.2, every point on the ray $\overrightarrow{AR}$ other than $A$ is in $h''$. Since $R$ is between $A$ and $B$, $B$ is on $AR$, and hence lies in $h''$, contrary to the hypothesis that $B$ is in $h$. Therefore it is impossible for $\overrightarrow{QR}$ and $\overrightarrow{QC}$ to be opposite rays, and we have $\overrightarrow{QR} = \overrightarrow{QC}$.

Now we are ready to prove that $C$ is in the interior of $\angle AQB$. Because $R$ is between $A$ and $B$, Theorem 5.1 asserts that $R$ is in interior of $\angle AQB$, and since $\overrightarrow{QR} = \overrightarrow{QC}$, we conclude that $C$ is also in this interior. [The reader is asked to supply the argument for this last step.]

[We remark that if one draws a figure for Theorem 6.2, it is hard to imagine how it could be false. We should remember, however, that our geometry is an idealization of our experience, and the fact that our axioms are sufficiently powerful to make a formal proof of this theorem possible at all encourages us to think that our geometry is not such a bad model of our perceptual geometry. The same comments apply to other theorems in this section, especially 6.3 and 6.4.]

Theorem 6.2 will be very useful when used in conjunction with Postulate 13.

The proof of the next theorem is quite complicated, and is easier to follow if we state, for reference, a simple consequence of Theorem 5.2.

*Lemma*.[*] If $\mathcal{L}$ is a line in plane p, U and V different points on $\mathcal{L}$, and X and Y points in p on opposite sides of $\mathcal{L}$, then $\overrightarrow{UX}$ and $\overrightarrow{VY}$ do not intersect.

(A "lemma" is a theorem which is of no real interest in itself but is useful in proving other theorems.)

*Theorem 6.3*. If P is a point in the interior of $\angle AQB$ then $\overrightarrow{QP}$ intersects $\overline{AB}$.

*Proof*: Let $\overrightarrow{QP'}$ and $\overrightarrow{QA'}$ be the rays opposite to $\overrightarrow{QP}$ and $\overrightarrow{QA}$. For convenience we list the various separation properties of our figure:

(i) P and A are on the same side of $\overleftrightarrow{QB}$,
(ii) P and B are on the same side of $\overleftrightarrow{QA}$,
(iii) P and P' are on opposite sides of $\overleftrightarrow{QA}$,
(iv) P and P' are on opposite sides of $\overleftrightarrow{QB}$,
(v) A and A' are on opposite sides of $\overleftrightarrow{QP}$,
(vi) A and A' are on opposite sides of $\overleftrightarrow{QB}$.

The first two follow from the definition of interior of an angle, the other four from the definition of opposite rays.

We wish to prove two things:

(1) That A and B are on opposite sides of $\overleftrightarrow{QP}$,
(2) That $\overline{AB}$ does not intersect $\overrightarrow{QP'}$.

Our theorem will follow from these, for if $\overline{AB}$ intersects $\overleftrightarrow{QP}$

(from (1)) and does not intersect $\overrightarrow{QP'}$, then $\overline{AB}$ must intersect $\overrightarrow{QP}$.

We prove (2) first, since it is easiest. From (ii) and (iii) it follows that B and P' are on opposite sides of $\overleftrightarrow{QA}$. If we apply the lemma, with $\mathcal{X} = \overleftrightarrow{QA}$, U = A, V = Q, X = B, Y = P', we see that $\overrightarrow{AB}$ and $\overrightarrow{QP'}$ do not intersect. By Theorem 4.9, $\overline{AB}$ and $\overrightarrow{QP'}$ do not intersect.

We now tackle (1), which is a much tougher job. The proof falls into three parts.

(a) From (i) and (vi), A' and P are on opposite sides of $\overleftrightarrow{QB}$. Applying the lemma with $\mathcal{X} = \overleftrightarrow{QB}$, U = B, V = Q, X = A', Y = P', we have that $\overrightarrow{BA'}$ does not intersect $\overrightarrow{QP'}$. Hence (Theorem 4.9), $\overline{BA'}$ does not intersect $\overrightarrow{QP'}$.

(b) This is similar to (a), starting with (ii) and (iii) and ending with $\overline{A'B}$ not intersecting $\overrightarrow{QP}$.

(c) From (a) and (b) it follows that $\overline{A'B}$ does not intersect $\overleftrightarrow{PQ}$; that is, A' and B are on the same side of $\overleftrightarrow{PQ}$. Combining this with (v) we have that A and B are on opposite sides of $\overleftrightarrow{PQ}$; that is, $\overline{AB}$ intersects $\overleftrightarrow{PQ}$, as was to be proved.

This theorem is a good illustration of the complications one can get into in proving an "obvious" result. This property of an angle is tacitly assumed in most geometry books, for instance when one speaks of the internal bisector of an angle of a triangle intersecting the opposite side (e.g. the first proof of Theorem 7.1).

The following theorem, while not as basic as Theorem 6.3, is interesting, It follows from Theorem 6.3, but is not easy.

*Theorem 6.4*.* If $P$ is a point in the interior of $\Delta ABC$ then any ray $\overrightarrow{PQ}$ intersects $\Delta ABC$ in exactly one point.

*Suggestion for proof*: Consider the possible positions of $Q$ with respect to the rays $\overrightarrow{PA}$, $\overrightarrow{PB}$, and $\overrightarrow{PC}$ and the angles they determine.

The following theorem will be used in Chapter 9. Its proof is relatively easy and is left as an exercise.

*Theorem 6.4a*.* Given $\angle BAC$, if $X$ and $U$ are points on $AB$ such that $0 < AX < AU$, and $Y$ and $V$ are points on $AC$ such that $0 < AV < AY$, then $\overline{XY}$ and $\overline{UV}$ intersect.

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## 3. Protractor Postulates.

*Postulate 11*. To every angle $ABC$ there corresponds a unique real number between 0 and 180, called the *measure* of angle and designated by $m(\angle ABC)$.

*Postulate 12*. Let $\overrightarrow{QX}$ be a ray on the edge of half-plane $h$. For any real number $r$ between 0 and 180 there is a point $Y$ in $h$ such that the $m(\angle XQY) = r$.

*Postulate 13*. If $D$ is a point in the interior of $\angle AQB$ then $m(\angle AQD) + m(\angle BQD) = m(AQB)$.

**Postulate 14**. If $\overrightarrow{QA}$ and $\overrightarrow{QB}$ are opposite rays and $\overrightarrow{QC}$ another ray then $m(\angle AQC) + m(\angle BQC) = 180$.

**Theorem 6.5**. Let $\overrightarrow{QA}$ be on the edge of half-plane $h$, and let $\overrightarrow{QB}$ and $\overrightarrow{QC}$ be different rays in $h$. Let $m(\angle AQB) = r$, $m(\angle AQC) = s$. Then

(1) $r \neq s$.

(2) $r < s$ if and only if $B$ is in the interior of $\angle AQC$.

(3) $m(\angle BQC) = |r - s|$.

Proof: From Theorem 6.2, either $B$ is in the interior of $\angle AQC$ or $C$ is in the interior of $\angle AQB$. We put the two cases in parallel columns.

| | |
|---|---|
| $B$ in interior of $\angle AQC$. | $C$ in interior of $\angle AQB$. |
| $m(\angle AQB) + m(\angle BQC) = m(\angle AQC)$. | $m(\angle AQC) + m(\angle BQC) = m(\angle AQB)$. |

(Postulate 13.)

| | |
|---|---|
| $m(\angle BQC) = m(\angle AQC) - m(\angle AQB)$ | $m(\angle BQC) = m(\angle AQB) - m(\angle AQC)$ |
| $= s - r$ | $= r - s$. |

(1) In either case, since $m(\angle BQC) \neq 0$ (Postulate 11) we have $r \neq s$.

(2) Since $r < s$ is equivalent to $s - r > 0$, and since $m(\angle BQC) > 0$, $r < s$ goes with the case, $B$ in the interior of $\angle AQC$.

(3) In either case, since $m(\angle BQC) > 0$, $m(\angle BQC) = |r - s|$.

This theorem shows that our angle measure behaves much like our distance measure as specified by Postulates 7 and 8.

*Theorem 6.6*.* If $\overrightarrow{QB}$ and $\overrightarrow{QC}$ are different rays in a half-plane with edge $\overleftrightarrow{QA}$ then we cannot have simultaneously both B in the interior of $\angle AQC$ and C in the interior of $\angle AQB$.

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4. *Theorems on Angles Formed by Two Intersecting Lines.* Consider two lines $\ell$ and $\ell'$ which intersect in the unique point Q. Then there exist points A and B on $\ell$ which lie on opposite sides of $\ell'$, and points A' and B' on $\ell'$ which lie on opposite sides of $\ell$. First of all we determine all possible angles with sides $\overrightarrow{QA}$, $\overrightarrow{QB}$, $\overrightarrow{QA'}$, $\overrightarrow{QB'}$. Because the sides of an angle must be non-collinear, there are exactly two angles with side $\overrightarrow{QA}$, namely,

$$\angle AQA', \quad \angle AQB'.$$

The angles with side $\overrightarrow{QB}$ are

$$\angle BQA', \quad \angle BQB';$$

with side $\overrightarrow{QA'}$,

$$\angle A'QA, \quad \angle A'QB;$$

and with side $\overrightarrow{QB'}$,

$$\angle B'QA, \quad \angle B'QB.$$

Of these there are exactly four distinct angles, namely

$$\angle AQA', \quad \angle AQB', \quad \angle BQA', \quad \angle BQB'.$$

These four angles are called *the angles formed by the intersecting lines* $\ell$ *and* $\ell'$. Of these we can select out exactly four pairs of adjacent angles, namely

$\angle AQA'$, $\angle AQB'$, common side $\overrightarrow{AQ}$,
$\angle BQA'$, $\angle BQB'$, common side $\overrightarrow{QB}$,
$\angle AQA'$, $\angle BQA'$, common side $\overrightarrow{QA'}$,
$\angle AQB'$, $\angle BQB'$, common side $\overrightarrow{QB'}$.

These are all possible pairs of adjacent angles, and we check that the first pair are actually adjacent in the sense of the definition, since, by assumption, $A'$ and $B'$ lie in opposite half-planes determined by the line $\ell$ containing the common side $\overrightarrow{QA}$. Because $\overrightarrow{QA}$ and $\overrightarrow{QB}$, and $\overrightarrow{QA'}$ and $\overrightarrow{QB'}$ are pairs of opposite rays, it is clear that there are exactly two pairs of vertical angles formed by $\ell$ and $\ell'$, namely

$$\angle AQA', \quad \angle BQB';$$

and

$$\angle AQB', \quad \angle BQA'.$$

Definitions. Two angles are congruent if their measures are equal; they are supplementary if the sum of their measures is 180 (each is said to be a supplement to the other).

Theorem 6.7. When two lines $\ell$ and $\ell'$ intersect, adjacent angles are supplementary, and vertical angles are congruent.

[Note that it is incorrect in our setup to say that vertical angles are equal.]

Proof: Let Q, A, B, A', B' be defined according to the beginning of this section. Although there are four pairs of adjacent angles it is sufficient to prove that $\angle AQA'$ and $\angle AQB'$ are supplementary, since any of the other pairs can then be taken care of by this case and a change of notation. Because the segment $\overline{A'B'}$ intersects $\ell$ in the unique point Q, Q is between A' and B', and $\overrightarrow{QA'}$ and $\overrightarrow{QB'}$ are opposite rays. By Postulate 14,

$$m(\angle AQA') + m(\angle AQB') = 180,$$

and the first assertion of the theorem is proved.

For the second assertion we shall prove that the vertical angles $\angle AQA'$ and $\angle BQB'$ are congruent. Consulting our table of adjacent angles, both of these angles are adjacent to the common $\angle AQB'$. Applying the first assertion of the theorem, the reader can now verify at once that $\angle AQA'$ and $\angle BQB'$ have equal measure.

*Theorem* *6.8**. If two adjacent angles formed by two intersecting lines are congruent, then all four are congruent.

*Definition*. If two adjacent angles formed by two intersecting lines are congruent the lines are said to be *perpendicular* and the angles are called *right* *angles*.

*Theorem* *6.9**. An angle is a right angle if and only if its measure is 90.

*Definitions*. An angle is said to be *acute* if its measure is less than 90; *obtuse* if its measure is greater than 90.

*Theorem* *6.10**. In a given plane there is one and only one line perpendicular to a given line at a given point on the line.

Theorem 6.11. If adjacent angles are supplementary their non-common sides are collinear.

Proof: Let $\angle AQB$ and $\angle BQC$ be adjacent and $m(\angle AQB) + m(\angle BQC) = 180$. Let $\overrightarrow{QA'}$ be opposite to $\overrightarrow{QA}$. Then $m(\angle AQB) + m(\angle BQA') = 180$ (Postulate 14), and so $m(\angle BQC) = m(\angle BQA')$. Now $A$ and $C$ are on opposite sides of $\overleftrightarrow{BQ}$ (definition of adjacent angles) and so are $A$ and $A'$. Hence $A'$ and $C$ are in the same half-plane with $\overleftrightarrow{BQ}$ as edge, and it follows from Theorem 6.5 (1) that $\overrightarrow{QC} = \overrightarrow{QA'}$. This proves the theorem.

## Exercise

Let $m$ and $n$ be distinct coplanar lines and $\ell$ a third line intersecting $m$ and $n$ in distinct points $A$ and $B$. ($\ell$ is a transversal to $m$ and $n$.) Give definitions of the following terms: interior angles, exterior angles, alternate interior angles, alternate exterior angles, corresponding angles (sometimes called exterior-interior angles).

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5. Theorems on Bisectors of Angles. In this concluding section we establish the existence and uniqueness of the internal and external bisectors of an angle.

Theorem 6.12. Given $\angle AQB$, there exists a unique ray $\overrightarrow{QM}$ in the interior of $\angle AQB$ such that

$$m(\angle AQM) = m(\angle BQM).$$

Proof: Let $h$ be the half-plane with edge $\overleftrightarrow{QA}$ and containing $B$. By Postulate 12 there is a ray $\overrightarrow{QM}$ in $h$ with $m(\angle AQM) = \frac{1}{2}m(\angle AQB)$. By Theorem 6.5 (2), $M$ is in the interior of $\angle AQB$. By Postulate 13 (or Theorem 6.5 (3)),

$$\begin{aligned} m(\angle BQM) &= m(\angle AQB) - m(\angle AQM) \\ &= \tfrac{1}{2}m(\angle AQB) \\ &= m(\angle AQM). \end{aligned}$$

Thus $\overrightarrow{QM}$ has the desired properties.

To show that there cannot be more than one such ray, let $\overrightarrow{QN}$ have the same properties. Then since

$$m(\angle AQN) = m(\angle BQN)$$

and

$$m(\angle AQN) + m(\angle BQN) = m(\angle AQB),$$

(Postulate 13) we must have

$$m(\angle AQN) = \tfrac{1}{2}m(\angle AQB) = m(\angle AQM).$$

By Theorem 6.5 (1) this is impossible if $\overrightarrow{QN}$ is different from $\overrightarrow{QM}$.

Definition. The ray $\overrightarrow{QM}$ described in Theorem 6.12 is called the bisector of $\angle AQB$, and is said to bisect $\angle AQB$.

Theorem 6.13.* Let $\overrightarrow{QM'}$ be the opposite ray to the bisector $\overrightarrow{QM}$ of $\angle AQB$. Then $m(\angle AQM') = m(\angle BQM')$.

Definition. $\overrightarrow{QM'}$ is called the external bisector of $\angle AQB$; by contrast $\overrightarrow{QM}$ is sometimes called the internal bisector of $\angle AQB$.

Theorem 6.14*. (Uniqueness of the external bisector.) Let $\overrightarrow{QP}$ lie in the exterior of $\angle AQB$, and suppose that $m(\angle AQP) = m(\angle BQP)$. Then $\overrightarrow{QP}$ coincides with the external bisector $\overrightarrow{QM'}$ of $\angle AQB$.

# Chapter 7

## Congruence

1. *Rigid Motion in Euclidean Geometry*. We have seen that the basic concepts of geometry are idealizations of portions of our physical experiences. One of the most basic of our physical experiences is the *motion of a rigid body*. Movement in general is one of the first physical phenomena we perceive. A clear distinction between the movement of a rope and that of a stick comes much later but is certainly well established by high school age.

Euclid took the concept of the motion of a rigid body for granted, just as he did most of the separation properties, and stated no postulates concerning it. If we wish to make use of the concept in our treatment we must introduce suitable definitions and postulates to handle it properly in our proofs. This can be done, but it turns out to be difficult and rather complicated.

It is also unnecessary, for if we examine Euclid's geometry we find that the concept of rigid motion is really not part of it. Euclid's is a static geometry, not a kinetic one. The kinetics of Euclidean space is of course very important, but its study is generally regarded as part of mechanics. This is purely a matter of convenience; we prefer, in the interest of simplicity, not to introduce *time* into our geometry. (It is interesting to note that in Einstein's Theory of Relativity it was found extremely useful to introduce the concept of time as part of the geometry.

This introduction was made in 1908 by the German mathematician Hermann Minkowski, and Minkowskian geometry is now a well established branch of mathematics.)

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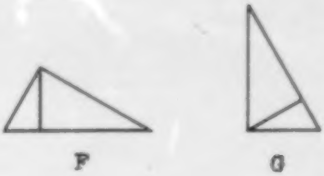
2. *Basic Definition of Congruence*. Consider Euclid's proof of the side-angle-side theorem. Although he speaks of placing one triangle on the other he is not really interested in the actual motion, involving all intermediate positions of the triangle, but only in the beginning and end positions. We shall therefore ignore the possible intermediate positions entirely in stating our definitions and postulates.



Given two figures (point sets) F and G; if, intuitively, G can be moved to coincide with F each point of G must be moved to coincide with a corresponding point of F, and, for the motion to be rigid, the distance between any two points of G must not change while G is being moved. How can we express the essential features of this situation without using the concept of motion? Very easily:

(1) There must be a one-to-one correspondence between the points of F and those of G;

(2) If P and Q are any two points of F, and P' and Q' the corresponding points of G, then we must have PQ = P'Q'.

A correspondence of this type shall be called a *congruence* between F and G, and the two figures shall be said to be *congruent*.

In most treatments of high school geometry the stress is put on the second of these definitions, as is easily seen from the wording of the theorems. We prefer to emphasize the first, for the following reasons. In certain cases two figures can be congruent in more than one way, that is, there may be more than one congruence between them. It may be of great importance to distinguish between these different congruences. In the second place, it is quite possible for a figure to be congruent to itself. In fact, this is always the case, since if every point is made to correspond to itself conditions (1) and (2) are certainly satisfied. But there are important cases when there is another, non-trivial, congruence of the figure with itself; that is, one in which not every point corresponds to itself. Such congruences are basic in a precise treatment of the notion of symmetry.

Notation. A one-to-one correspondence in which A corresponds to P, B to Q, C to R, etc. shall be written

$$A, B, C \ldots \longleftrightarrow P, Q, R \ldots .$$

If this is a congruence we write

$$A, B, C \ldots \cong P, Q, R \ldots .$$

Note that

$$A, B, C \cong P, Q, R$$

is the same thing as

$$A, C, B \cong P, R, Q$$

but not the same as

$$A, C, B \cong P, Q, R.$$

Sample theorem. If $AB = XY$ there is a congruence between $\overline{AB}$ and $\overline{XY}$ such that $A, B \cong X, Y$.

Partial proof: Set up a coordinate system on $\overleftrightarrow{AB}$ with $A$ having coordinate zero and $B$ a positive coordinate. Set up a coordinate system on $\overleftrightarrow{XY}$ with $X$ having coordinate zero and $Y$ a positive coordinate. Let a point $P$ on $\overleftrightarrow{AB}$ and a point $Q$ on $\overleftrightarrow{XY}$ correspond if they have the same coordinates in the respective systems. This correspondence is then a congruence (proof left to reader).

Corollary. There is a non-trivial congruence of a segment with itself; in particular, such that $A, B \cong B, A$.

Proof: Merely take the special case $X = B$, $Y = A$, which was not excluded.

The ambitious reader may try his hand at the proof of the following theorem:

Two congruences as described above, with $A, B \cong X, Y$ and $A, B \cong Y, X$, are the only congruences between $\overline{AB}$ and $\overline{XY}$.

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3. A More Suitable Definition of Congruence. Having now given a satisfactory definition of congruence we must confess that it is not suitable for a development of high school geometry. One trouble lies in the difficulty encountered in relating it to the measure of angles. We certainly want angle measure to be preserved also in our one-to-one correspondence. We can take care of this with an appropriate postulate, but the treatment is somewhat

artifical. Then too, we constantly run into proofs of the type given above, requiring us to set up coordinate systems on a line.

To avoid these complications we modify our approach and do not try to give a general definition of congruence that holds for all figures. Instead we define the notion in a series of steps, as follows.

Definitions. Two line segments are said to be congruent if they have the same length. Two angles are said to be congruent if they have the same measure.

Two triangles are said to be congruent if there is a one-to-one correspondence between their vertices such that corresponding sides have the same length and corresponding angles have the same measure. In this case the correspondence is called a congruence

The big difference between this last definition and the one given in Section 2 is that this one is stated in terms of only a finite number of points, the six vertices of the triangles, whereas the other involved all the points of the two figures. This one is therefore easier to apply. Its application is all we ordinarily need in developing Euclidean geometry.

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4. The Congruence Postulate. Having now found, in the concept of "congruence," a way to avoid talking about "motion" we must introduce some of the intuitive properties of rigid motion as postulates concerning congruence. It turns out that only one postulate is needed. There are several choices for this,

but a simple and intuitive one to choose is the side-angle-side statement itself:

*Postulate 15*. If there is a one-to-one correspondence between the vertices of two triangles such that two sides and the included angle of one triangle are congruent to the corresponding parts of the other then the correspondence is a congruence.

If one needs a justification for introducing a postulate the familiar "proof" of Euclid is an intuitive argument that this statement agrees with our conception of geometry as an idealization of the physical world.

Let us now look at the proofs of a few basic theorems.

*Theorem 7.1*. If two sides of a triangle are congruent the angles opposite these are congruent.

*First proof*: In $\triangle ABC$ let $AC = BC$. Let $\overrightarrow{CP}$, the bisector of $\angle ACB$, intersect $AB$ in $D$. In the correspondence $ACD \rightarrow BCD$ we have $AC = BC$, $CD = CD$, $m(\angle ACD) = m(\angle BCD)$. Hence by the postulate $\triangle ACD \cong \triangle BCD$ and $m(\angle DAC) = m(\angle DBC)$.

This is the proof given in most high school geometries. It makes use of two fairly complicated earlier theorems; namely, the existence of an angle bisector and the fact that ray $\overrightarrow{CP}$ intersects segment $\overline{AB}$. (Theorems 6.12 and 6.3.)

*Second proof*: In $\triangle ABC$ let $AC = BC$. In the correspondence $ABC \leftrightarrow BAC$ we have $AB = BA$, $AC = BC$, $m(\angle ACB) = m(\angle BCA)$. Hence the correspondence is a congruence, and $m(\angle BAC) = m(\angle ABC)$.

In the second proof we make use of a non-trivial congruence of the figure with itself.

In physical terms the first proof can be said to fold the triangle so as to make the two halves coincide. The second proof turns the whole triangle over. Proofs of this kind are frequently applicable when the figure has an axis of symmetry.

Theorem 7.2. Through a point $P$ not on a line $m$ there passes a line perpendicular to $m$.

Proof: Let $A$ and $B$ be any two different points of $m$. $m$ and $p$ determine a plane, which is separated into two half-planes by $m$. In the half-plane not containing $P$ take $\overrightarrow{AR}$ so that $m(\angle BAR) = m(\angle BAP)$. On $\overrightarrow{AR}$ take point $Q$ so that $AQ = AP$. Since $Q$ and $P$ are on opposite sides of $m$, $\overline{PQ}$ intersects $m$ in a point $D$.

Case 1. $D = A$. By the definition of perpendicularity, $PQ$ is perpendicular to $m$.

Case 2. $D \neq A$, $D$ on ray $\overrightarrow{AB}$. Show that the correspondence $PAD \leftrightarrow QAD$ is a congruence and that this proves the theorem.

Case 3. $D \neq A$, $D$ on ray opposite to $\overrightarrow{AB}$. Proof left to reader.

This proof, though long, avoids the difficulty in Euclid's proof of having to assume something about the intersection of a line and a circle.

Theorem 7.3. If two triangles are in one-to-one correspondence such that two angles and the included side of one are congruent to the corresponding parts of the other then the correspondence is a congruence.

Proof: Let $ABC \leftrightarrow XYZ$ so that $m(\angle A) = m(\angle X)$, $m(\angle B) = m(\angle Y)$, $AB = XY$. On ray $\overrightarrow{AC}$ take a point $D$ so that $AD = XZ$. The correspondence $ABD \leftrightarrow XYZ$ is then a congruence, and so $m(\angle ABD) = m(\angle XYZ) = m(\angle ABC)$. It follows that $\overrightarrow{BD} = \overrightarrow{BC}$, and hence $D = C$. Therefore $\Delta ABC \cong \Delta XYZ$. (The reader is expected to fill in the details of this and similar proofs.)

This type of proof might be called "proof by identification." The essential feature is the construction of a figure which has the desired property ($\Delta ABD$ in this case) followed by the demonstration that this figure is identical with the given one. For another proof by identification consider the side-side-side theorem.

Theorem 7.4. If two triangles are in one-to-one correspondence such that the three sides of one are congruent to the corresponding sides of the other then the correspondence is a congruence.

Proof: Let $ABC \leftrightarrow XYZ$. In the half-plane with edge $\overleftrightarrow{AC}$ containing $B$ take $P$ such that $m(\angle CAP) = m(\angle X)$, and on $\overrightarrow{AP}$ take $D$ such that $AD = XY$. Then $\Delta ADC \cong \Delta XYZ$, and $CD = XY = CB$. We wish to show that $D = B$. Suppose that $D \neq B$, and let $M$ be the mid-point of $\overline{BD}$. Then $\Delta BMA \cong \Delta DMA$. and so $\overleftrightarrow{AM}$ is perpendicular to $\overleftrightarrow{BD}$. In exactly the same way we can show that $\overleftrightarrow{CM}$ is perpendicular to $\overleftrightarrow{BD}$. This would mean that line $\overleftrightarrow{AC}$ contains $M$, which is impossible. (If M lies on $\overleftrightarrow{AC}$ then B and D are on opposite sides of $\overleftrightarrow{AC}$, contrary to construction of D.) Hence $D = B$ and the theorem is proved.

This proof has some features in common with Euclid's Proposition 7, Book I. He proves $D = B$ by a rather elaborate argument with angles.

It is interesting at this point to analyze the logical structure of the proof of Theorem 7.4. It is surprisingly complicated.

(1) Construct $\Delta$ ADC, with D and B on the same side of $\overleftrightarrow{AC}$.

(2) Direct proof that $\Delta ADC \cong \Delta XYZ$.

(3) Indirect proof that $D = B$.

  (a) Assume $D \neq B$.

  (b) Use of previous theorem to justify existence of mid-point M.

  (c) Indirect proof that BMA and DMA are triangles.

    (i) Assume A lies on line $\overleftrightarrow{BMD}$.

    (ii) Indirect proof that A does not lie between B and D.

      ($\alpha$) Assume A is between B and D.

      ($\beta$) $\overline{BD}$ intersects line $\overleftrightarrow{AC}$ in A.

      ($\gamma$) Contradicts (1)

    (iii) Direct proof that $D = B$.

    (iv) Contradicts (a)

  (d) Direct proof that $\Delta BMA \cong \Delta DMA$.

  (e) Direct proof that $\overleftrightarrow{AM}$ is perpendicular to $\overleftrightarrow{BD}$.

  (f) Proof that $\overleftrightarrow{CM}$ is perpendicular to $\overleftrightarrow{BD}$, by analogy with (c), (d), (e).

(g) Direct proof that $\overline{BD}$ intersects $\overleftrightarrow{AC}$.

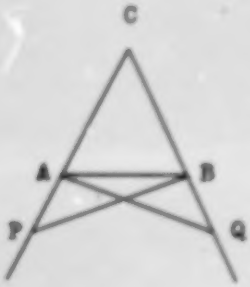
(h) Contradicts (1)

(4) Identity of $\triangle$ ABC and $\triangle$ ADC.

(5) Theorem follows from (2).

Note that most of the complication comes in proving the "obvious" step (c). Such complications are of course not to be emphasized in a high school course.

### Exercises



1. Prove Theorem 7.1 by Euclid's method, illustrated in the adjacent figure.

2. Carry out Euclid's proof of his Proposition 7, Book I, in the framework of our set of postulates.

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5. *Further Theorems*. With the three basic congruence theorems proved we can move along the regular sequence of theorems, with only occasional modifications needed to insure logical rigor. We shall list the more important theorems, leaving most of the proofs to the reader.

*Theorem 7.5*$^*$. If two angles of a triangle are congruent the sides opposite them are congruent and the triangle is isosceles.

*Theorem 7.6*$^*$. In a given plane p, the set of points equidistant from two given points A and B is the line perpendicular to $\overline{AB}$ at its mid-point.

Theorem 7.7. If a line $\ell$ is perpendicular to each of two distinct lines m and n at their point of intersection W, then it is perpendicular to every line containing W and coplanar with m and n.

Proof: Let p be the plane containing m and n, and let r be any line in p and containing W. We wish to prove that r is perpendicular to $\ell$. If $r = m$ or $r = n$ this follows by assumption, so we have only to consider the case in which r, m, and n are all different. Let h be one of the half-planes with edge m, let $\overrightarrow{WP}$ and $\overrightarrow{WQ}$ be the rays of r and n which lie in h, and let S be any point of m distinct from W. By Theorem 6.2, either Q is in the interior of $\angle SWP$ or P is in the interior of $\angle SWQ$. If the former, then by Theorem 6.3 $\overrightarrow{WQ}$ intersects $\overline{PS}$; if the latter, then $\overrightarrow{WP}$ intersects $\overline{QS}$. In either case we obtain a line not containing W and intersecting m, n, and r. The standard proof of this theorem can now be carried out.

Theorem 7.8. All the lines perpendicular to a given line at a given point lie in one plane.

Proof: Let $\ell$ be a line and W a point of $\ell$. Let Q be a point not on $\ell$ (Postulate 1), u the plane containing Q and $\ell$ (Theorem 3.3), and m the line in u perpendicular to $\ell$ at W (Theorem 6.10). Let R be a point not in u, v the plane containing R and $\ell$, and n the line in v perpendicular to $\ell$ at W. If $m = n$ then u and v each contain the intersecting lines $\ell$ and m, and so coincide (Theorem 3.4). This is impossible since v contains R and u does not. Hence $m \neq n$. Let p be the plane containing m and n.

We wish to show that if k is any line perpendicular to $\ell$ at W then k lies in p. If k is such a line let t be the plane containing $\ell$ and k. If $t = p$ lines $\ell$, m, and n would be coplaner, which is impossible by Theorem 6.10. Hence t and p are distinct planes which intersect at W, and so by Postulate 5 they have a line k' in common. By Theorem 7.7, k' is perpendicular to $\ell$ at W, and so by Theorem 6.10 $k' = k$. Thus k lies in p.

Note that the basic idea of this proof is very simple. The length of the proof is due to all the little details that must be filled in to make a logical sequence of steps. In most proofs we omit these details, just as we omit continual references to the associative, commutative, and distributive laws when doing algebra. We should be aware of their existence, however, and be able to fill them in if required.

Definition. The plane which contains all lines perpendicular to a given line at a given point of the line is said to be perpendicular to the line at the point.

Theorem 7.9*. The set of points equidistant from two given points A and B is the plane perpendicular to $\overline{AB}$ at its midpoint.

The next group of theorems deals with inequalities between measures of angles and of distances. It will be convenient to shorten our terminology and say "angle A is greater than angle B" or "$\overline{AB}$ is greater than $\overline{CD}$" instead of "the measure of angle A is greater than the measure of angle B" or "the length of $\overline{AB}$ is greater than the length of $\overline{CD}$."

Theorem 7.11. An exterior angle of a triangle is greater than either of the non-adjacent interior angles.

Proof: Given $\Delta ABC$, let $\overrightarrow{BD}$ be the ray opposite to $\overrightarrow{BA}$. Let E be the mid-point of $\overline{BC}$. On ray $\overrightarrow{EF}$ opposite to $\overrightarrow{EA}$ take F such that $EF = EA$. By Theorem 6.7 $m(\angle BEA) = m(\angle CEF)$. It follows from Postulate 15 that $\Delta BEF \cong \Delta CEA$, and so $m(\angle C) = m(EBF)$. Now from Theorem 5.3 it follows that F is in the interior of $\angle DBC$. (This was the reason for proving Theorem 5.3.) Hence from Postulate 13

$$\begin{aligned} m(\angle DBC) &= m(\angle DBF) + m(\angle FBC) \\ &= m(\angle DBF) + m(\angle C) \\ &> m(\angle C). \end{aligned}$$

Similarly one can prove $m(\angle DBC) < m(\angle A)$, and the theorem is established.

Theorem 7.12.* If two sides of a triangle are not congruent the angles opposite them are not congruent, and the greater side is opposite the greater angle.

Theorem 7.13.* If two angles of a triangle are not congruent the sides opposite them are not congruent, and the greater angle is opposite the greater side.

Theorem 7.14*. The sum of the measures of any two sides of a triangle is greater than the measure of the third side.

Theorem 7.15*. Through a given point not on a given line there passes only one line, and only one plane, perpendicular to the given line.

Theorem 7.16*. If $P$ is a point not on a line $\ell$ and $A$ and $B$ two different points of $\ell$ such that line $\overleftrightarrow{AP}$ is perpendicular to $\ell$, then $\overline{AP}$ is less than $\overline{BP}$.

## Chapter 8

## Parallelism

### 1. Existence Theorems.

Definition. Two coplanar non-intersecting lines are said to be parallel.

We indicate that lines $m$ and $n$ are parallel by writing $m \parallel n$.

Throughout this chapter we shall consider only points and lines lying in one given plane. This will simplify our language while still enabling us to discuss the basic ideas. Extensions of these ideas to parallelism of lines and planes in space can be done in the conventional way.

With our present set of postulates we can easily prove the existence of parallels. It is convenient to give a few more definitions.

Definition. A line $\ell$ is a transversal to two lines $m$ and $n$ if it intersects them in two distinct points.

Definition. Let a transversal $\ell$ intersect line $m$ in point $P$ and line $n$ in point $Q$. Let $A$ be a point on $m$ and $B$ a point on $n$ so that $A$ and $B$ are on opposite sides of $\ell$. Then $\angle APQ$ and $\angle BQP$ are alternate interior angles.

We can now state the basic theorem as follows.

Theorem 8.1. If two lines are intersected by a transversal so that a pair of alternate interior angles are congruent, then the two lines are parallel.

Proof: Let $\ell$, $m$, $n$, $P$, $Q$, $A$, $B$ be as in the definition of alternate interior angles and let $\angle APQ \cong \angle BQP$. Assume that $m$ and $n$ are not parallel. Then they intersect in some point $R$. Since $A$ and $B$ are on opposite sides of $\ell$, either $A$ or $B$ is on the same side of $\ell$ as $R$. Suppose that $R$ and $A$ are on the same side of $\ell$. In the triangle $PQR$, $\angle APQ = \angle RPQ$ is an angle of the triangle and $\angle BQP$ is an opposite exterior angle. Hence by Theorem 7.11, $m\angle BPQ > m\angle APQ$. But this contradicts the hypothesis that $\angle BPQ \cong \angle APQ$. Hence $m$ and $n$ cannot intersect on the same side of $\ell$ as $A$. An exactly similar argument shows that they cannot intersect on the same side of $\ell$ as $B$. Hence they cannot intersect at all, and so they are parallel.

Corollary 8.1. Two perpendiculars to the same line are parallel.

Corollary 8.2. Through a given external point there is at least one line parallel to a given line.

## Exercises

1. Define alternate exterior angles, and corresponding angles (sometimes called exterior-interior angles). State and

prove theorems analogous to Theorem 8.1 for alternate exterior angles and for corresponding angles.

2. Prove Corollary 8.1 independently of Theorem 8.1 and without using Theorem 7.11. [First prove that only one perpendicular can be drawn to a line from an external point.]

---

2. *The Parallel Postulate*. Corollary 8.2 guarantees the existance of at least one parallel to a line through an external point. The question then arises: "Can there be more than one such parallel?" Our intuition say "No", so we try to prove this on the basis of our postulates. Unfortunately this cannot be done (this point will be discussed later) so if we wish to have our geometry match our intuition we must introduce a new postulate. This we now do.

*Postulate 16*. Through a given external point there is at most one line parallel to a given line.

A vast array of familiar theorems follows from this postulate. We can mention only a few of the most important and interesting ones here.

Theorem 8.2*. (Converse of Theorem 8.1.) If two parallel lines are intersected by a transversal any pair of alternate interior angles are congruent.

*Theorem 8.3*.* The sum of the measures of the angles of a triangle is 180.

*Definitions*.$^*$ *Parallelograms*, *trapezoid*, etc.

*Theorem 8.4*.$^*$ Either diagonal of a parallelogram separates it into two congruent triangles.

*Theorem 8.5*.$^*$ If a pair of opposite sides of a quadrilateral are congruent and parallel then the other pair of opposite sides are congruent and parallel.

*Theorem 8.6*.$^*$ The diagonals of a parallelogram bisect each other.

The difficult part of the proof of this theorem lies in proving that the diagonals intersect. This can be shown by making use of Theorem 6.3.

*Theorem 8.7*.$^*$ The segment whose end-points are the mid-points of two sides of a triangle is parallel to the third side and half as long.

*Theorem 8.8*.$^*$ The medians of a triangle are concurrent.

*Theorem 8.9*.$^*$ If a set of parallel lines intercept congruent segments on one transversal, then they intercept congruent segments on any transversal.

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3. *The Role of the Parallel Postulate*. Since intuition is notoriously unreliable one may well question the advisability of introducing the Parallel Postulate. Why not just go ahead as we have been doing on the basis of our fifteen postulates, proving theorems without making any general stipulation about the number

of parallels through an external point? Of course we can try this, but if we do we find that the properties of parallels are so important that theorems group themselves more or less automatically into three categories:

Type 1. Theorems independent of the number of parallels through an external point;

Type 2. Theorems whose proof requires that there be only one parallel through an external point;

Type 3. Theorems whose proof requires that there be more than one parallel through an external point.

The collection of theorems of Type 1 is sometimes called Neutral Geometry. It includes all the theorems we proved before introducing the Parallel Postulate, as well as many others, of course. On the whole, however, the number of interesting theorems of Type 1 is small compared with the number of either of the other types.

The theorems of Types 1 and 2, taken together, constitute Euclidean Geometry.

The theorems of Types 1 and 3 constitute Lobachevskian[1] geometry. To develope this geometry we would replace Postulate 16 by

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[1] N. I. Lobachevski (1793-1856) was one of three men who independently developed this geometry. The other two, K. F. Gauss and J. Bolyai, did equally good work but were slow in publishing their results and consequently attracted less attention.

Postulate 16'. Through a given external point there are at least two distinct lines parallel to a given line.

Lobachevskian geometry naturally has radical differences from Euclidean geometry. Here, for example, are two striking theorems and a definition in Lobachevskian geometry.

Theorem. The sum of the measures of the angles of a triangle is less than 180.

Definition. The deficiency of a triangle is the difference between the sum of the measures of the angles of the triangle and 180.

Theorem. The areas of triangles are proportional to their deficiencies.

There is nothing in Euclidean geometry analogous to this last theorem.

## Exercise

Prove that in Lobachevskian geometry there is an infinite number of parallels to a given line through an external point.

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In section 2 we remarked on the possibility of proving Postulate 16 from the earlier postulates. If this could be done then Lobachevskian geometry, since it involves the negation of

Postulate 16, would be inconsistent. Similarly If we could prove Postulate 16' from the earlier ones then Euclidean geometry would be inconsistent. Probably nobody ever suspected the latter to be the case, but there were many doubts about the consistency of any other kind of geometry. Finally, in 1866, it was proved that Lobachevskian and Euclidean geometries were equally consistent; any contradiction in one would necessarily imply a contradiction in the other.

Why, then, do we commonly study Euclidean geometry to the neglect of Lobachevskian? Partly because of tradition, but primarily because Euclidean geometry is simpler, richer in theorems, and more easily adaptable to the representation of physical phenomena.

Lobachevskian geometry can suitably be called non-Euclidean geometry. However, this last term embraces still another type of geometry, Riemannian Geometry, which differs still more radically from Euclidean. In addition to dropping Postulate 16 we throw out the Separation Postulates and radically change the Ruler Postulate. The end result is that we lose many of those theorems of neutral geometry that depend on separation properties; in particular we lose Theorem 7.11 and we can no longer prove the existence theorem Corollary 8.2. It is thus possible to introduce

Postulate 16". There are no parallel lines.

This leads to Riemannian geometry.

Actually, the loss of the separation properties and the ruler postulate are not as serious as might be imagined, and

Riemannian geometry bears close resemblances to Lobachevskian. Thus the two theorems and the definition given above can be translated into Riemannian geometry by merely replacing "less" by "greater" and "deficiency" by "excess".

## Chapter 9

## Area

1. Informal Properties of Area. At an early age we all became familiar with certain properties of figures that involve a concept of "area". We easily accept that a 4" x 6" rectangle of paper contains "more paper" than a 2" x 3" rectangle. It takes a little experience to teach us that the former rectangle contains four times as much paper as the latter. Still more experience, involving further cutting of the paper, is needed to show that the 4" x 6" rectangle contains just as much paper as a 3" x 8" one, or as a $2\frac{2}{3}$" by 9" one.

The concept of the "area" of a "plane region" is an abstraction of this paper cutting process. Instead of actually cutting paper we make geometric dissections of the regions and use our knowledge of geometry to make the comparisons. This was Euclid's approach, and he regards it as so natural that he never makes any comment on the notion of area. In a modern treatment we must of course either define "area" or introduce it as an undefined term and specify some postulates concerning it. A definition of area in terms of the concepts we have already introduced turns out to be possible, but it is a very long process, involving ideas and methods entirely out of keeping with elementary mathematics. We shall adopt the second approach, essentially introducing area as an undefined term and postulating enough simple properties to enable us to develop a suitable theory. This, after all, was the way we introduced the concepts of distance and congruence.

In another respect our treatment of area will differ from Euclid's. Since his mathematics had no well-developed number system to use he was forced to word his theorems in terms of comparison of areas. We, on the other hand, can use the same scheme we adopted in handling distances - we can choose a "unit of area" and express all areas as real number multiples of this unit. Thus for us "area" is a real number attached to a "plane region" and satisfying certain well-chosen properties.

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2. The Area Postulates. In the above discussion the term "plane region" was deliberately left vague. It turns out that a general definition of this term is, like a definition of "area", a matter for advanced mathematics. We shall therefore limit our discussion here to a special type of region defined as follows.

Definitions. A triangular region is the union of a triangle and its interior. Two triangular regions are said to be non-overlapping if their intersection is either the empty set, a point, or a segment.

Definitions. A polygonal region is the union of a finite number of coplanar, non-overlapping triangular regions. Two polygonal regions are said to be non-overlapping if their intersection is either empty or consists of a finite number of points or lines or both.

For the rest of this chapter the word "region" shall always signify "polygonal region". Also, we shall frequently speak of "the area of a triangle" or "the area of a rectangle", meaning thereby the areas of the regions consisting of these figures and their interiors. We shall designate the area of a region R simply by "area R".

Postulate 17. With every polygonal region there is associated a unique positive real number, called the area of the region.

In Section 1 we talked about comparing areas of regions by cutting up the regions and comparing the pieces (presumably by moving them to see if they could be made to coincide). This "moving around" implies that area must not change under rigid motion, and hence motivates the next postulate.

Postulate 18. If two triangles are congruent the triangular regions have the same area.

The cutting up process of determining area also implies the following property.

Postulate 19. If a region R is the union of two non-overlapping regions S and T then

$$\text{area } R = \text{area } S + \text{area } T.$$

These three postulates would enable us to develop a theory of area, but they suffer from one defect - they establish no connection between the unit of area and the unit of distance. It is highly convenient to have such a connection, and this can be established by taking the area of some conveniently sized and shaped region as a unit. It is customary to take as this region a square whose edge has unit length, and this we shall do.

If we simply postulate this much, however, we still find it a difficult task to prove, for all cases, the basic formulas for the areas of plane figures in terms of measurements of distance. To see the reason for this consider some rectangular regions. A $2 \times 3$ region is the union of six non-overlapping unit squares and so has area $6$. A $2 \times 3.16$ region is the union of $63{,}200$ non-overlapping squares of side $.01$; since the unit square is the union of $10{,}000$ such squares each of the small squares has area $.0001$, and the $2 \times 3.16$ rectangle has an area of $6.32$. However this method breaks down for a $2 \times \sqrt{10}$ rectangle since $\sqrt{10}$ is an irrational number. This is the so-called "incommensurable case" that has caused so much trouble in past treatments of elementary geometry. We choose to avoid this trouble by making a stronger statement for our fourth and final area postulate.

Definition. By Theorem 8.4 a diagonal of a rectangle separates it into two triangles. The corresponding triangular regions are non-overlapping and their union is called a rectangular region.

Postulate 20. The area of a rectangular region is the product of the lengths of two adjacent sides.

## Exercises.

1. Prove that two triangular regions are non-overlapping if and only if the interiors of the two triangles do not intersect.

2. Show that to each rectangle there corresponds a unique rectangular region by proving the following:

If ABCD is a rectangle then the union of the two triangular regions ABC and ADC is the same as the union of the two triangular regions ABD and CBD.

Note: To show that two point sets are equal we must show that any point lying in either one of them lies in the other. Thus the required proof will consist in showing that

(i) If P is in either of the regions ABC or ADC then P is in at least one of ABD or CBD;

(ii) If P is in either of ABD or CBD then P is in at least one of ABC or ADC.

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### 3. Areas of Polygonal Regions.

Definitions. Any side of a triangle or a parallogram, or either of the two parallel sides of a trapezoid, may be called a base of the figure. An altitude corresponding to a given base is the segment perpendicular to the base from a vertex not lying on the base. The legs of a right triangle are the two sides adjacent to the right angle; the hypotenuse is the remaining side.

In the following theorems we shall be concerned with relations between areas and lengths of bases, altitudes, etc. It is customary to abbreviate the phrase "length of ______" to simply "______". This double use of a word such as "leg" to mean both a segment and the length of that segment could cause confusion but rarely does, since the proper meaning is always evident from the context.

Theorem 9.1. The area of a right triangle is half the product of its legs.

Proof: Let ABC be a right triangle with right angle at C. Let L be the line through A parallel to $\overleftrightarrow{BC}$ and M the line through B parallel to $\overleftrightarrow{AC}$. $L \perp \overleftrightarrow{AC}$ and $M \parallel \overleftrightarrow{AC}$ so L and M cannot be parallel, and they therefore intersect in a point D. ACBD is a rectangle (opposite sides are parallel and $\angle C$ is a right angle) and ABC is one of the two triangular regions whose union is the rectangular region. By Theorem 8.4 and Postulates 18, 19 and 20,

$$\text{area } \triangle ABC = \frac{1}{2} \text{ area } ACBD = \frac{1}{2}AC \cdot BC.$$

Theorem 9.2.* The area of a triangle is half the product of any side and the corresponding altitude.

Sketch of proof: In $\triangle ABC$, if h is the altitude upon side $\overline{AB}$, then either

(1) $\triangle ABC$ is a right triangle with legs h and AB; or

(2) Region ABC is the non-overlapping union of two right triangles with a common leg h and with the sum of the other two legs equalling AB; or

(3) Union of region ABC and a suitable right triangle with leg h is another right triangle with leg h, the difference of the other legs of these two right triangles being AB.

In the last two cases the theorem follows from Theorem 9.1 and Postulate 19.

Definitions.* Area of a parallelogram; of a trapezoid.

Theorem 9.3.* The area of a parallelogram is the product of a base and the corresponding altitude.

Theorem 9.4.* Area of a trapezoid.

The following corollary of Theorem 9.2 will be useful later.

Corollary 9.1.* (a) If two triangles have equal altitudes the ratio of their areas equals the ratio of their bases.

(b) If two triangles have equal bases the ratio of their areas equals the ratio of their altitudes.

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4. Two Basic Theorems. The more elaborate aspects of Euclidean geometry rest upon two theorems which involve the connection of distance with perpendiculars and with parallels, respectively. The first of these is the most important theorem (if such a term can be applied to a single piece of a deductive theory) in all geometry, the Pythagorean Theorem. The second is the basis of the theory of similar figures. Together they form the foundations of coordinate geometry, whioh in turn serves as the start of much modern mathematics.

These two theorems follow from our theory of area, and indeed this was essentially the way Euclid proved them.

Theorem 9.5. In any right triangle the square of the hypotenuse is equal to the sum of the squares of the legs.

Sketch of proof: Let $T$ be a right triangle with legs $a$, $b$ and hypotenuse $c$. Let $ABCD$ be a square of side $a + b$, and let $W, X, Y, Z$ be points on the square such that

$$AW = BX = CY = DZ = a, \quad WB = XC = YD = ZA = b.$$

Then each of the triangles AWZ, BXW, CYX, DZY is congruent to T, and it follows that WXYZ is a square of side c. Since the large square is the non-overlapping union of the small square and the four triangles we have

$$(a + b)^2 = c^2 + 4(\tfrac{1}{2}ab),$$

from which follows

$$a^2 + b^2 = c^2.$$

The missing part of this proof consists in showing that the large region is the union of the five smaller ones. Actually the situation is even worse, since a square region is defined in terms of two triangular ones. We really have to prove the following:

Given the eight points A, B, C, D, W, X, Y, Z as described above, consider the two sets of triangles

(I) AWZ, BXW, CYX, DZY, WXZ, YXZ;
(II) ABC, ADC.

(a) Prove that a point in the interior of any triangle of (I) is not in the interior of any other triangle of (I). [This proves the non-overlapping of the regions.]

(b) Prove that

(i) If a point is in any triangular region of (I) then it is in one of the triangular regions of (II); and conversely

(ii) If a point is in either of the triangular regions of (II) then it is in one of the triangular regions of (I).

[This proves that the two unions are the same.]

These proofs simply involve an enormous amount of fussing around with the separation properties of Chapters 5 and 6. This is the sort of proof which one dismisses with the comment, "The method is obvious and the details are boring."

There are many other similar proofs of the Pythagorean Theorem but they all have roughly the same characteristics. A different type of proof will be given later.

Theorem 9.6. If two transversals divide the sides of an angle proportionately then the transversals are parallel. More specifically, if $B$ and $D$ are distinct points on one side of $\angle A$, and $C$ and $E$ points on the other side, such that $AB/AD = AC/AE$, then $\overleftrightarrow{BC} \parallel \overleftrightarrow{DE}$.

Proof: Since $B$ and $D$ are distinct either $AB/AD < 1$ or $AB/AD > 1$. We shall treat the first case; the other can be handled similarly by merely interchanging the roles of $B$ and $C$, and $D$ and $E$. We first show that $\overline{DE}$ does not intersect $\overleftrightarrow{BC}$. For since $AB < AD$, $A$ and $D$ are on opposite sides of $\overleftrightarrow{BC}$, and similarly $A$ and $E$ are on opposite sides of $\overleftrightarrow{BC}$. Hence $D$ and $E$ are on the same side of $\overleftrightarrow{BC}$.

We next apply Corollary 9.1 (a), to get

$$\frac{\text{area } BEA}{\text{area } DEA} = \frac{AB}{AD}, \qquad \frac{\text{area } DCA}{\text{area } DEA} = \frac{AC}{AE}.$$

Since we are given that $AB/AD = AC/AE$ we thereby obtain

$$\text{area } BEA = \text{area } DCA.$$

Since $E$ and $A$ are on opposite sides of $\overleftrightarrow{BC}$ the interiors of

$\Delta$ABC and $\Delta$ EBC cannot intersect, and so region BEA is the non-overlapping union of ABC and EBC. Similarly DCA is the non-overlapping union of ABC and DBC. Hence by Postulate 19 the above equation becomes

$$\text{area ABC} + \text{area EBC} = \text{area ABC} + \text{area DBC},$$

from which follows

$$\text{area EBC} = \text{area DBC}.$$

Now $\Delta$ EBC and $\Delta$ DBC have the same base BC, and so by Corollary 9.1 (b) they have equal altitudes. That is, if F and G are the feet of perpendiculars from D and E onto $\overleftrightarrow{BC}$, then DF = EG. Since, as was shown above, $\overline{DE}$ does not intersect $\overleftrightarrow{BC}$, and hence does not intersect $\overline{FG}$, DEGF is a quadrilateral. $\overline{DF}$ and $\overline{EG}$ are congruent and parallel, and so by Theorem 8.5, $\overleftrightarrow{DE}$ and $\overleftrightarrow{GF}$ are parallel, which proves the theorem.

The converse of this theorem, which is equally important in applications, can be given an independent proof essentially by reversing the steps in the above argument. There are a few added difficulties, however, (for instance, if we are given that B is between A and D and that $\overleftrightarrow{BC} \parallel \overleftrightarrow{DE}$, we must prove that C is between A and E, for which see Theorem 6.4a) and so it is easier to use Theorem 9.6 to prove its converse.

*Theorem 9.7*. Parallel transversals divide the sides of an angle proportionately. More specifically, if B and D are distinct points on one side of $\angle$ A, and C and D points on the other side, such that $\overleftrightarrow{BC} \parallel \overleftrightarrow{DE}$, then AB/AD = AC/AE.

Proof: On $\overrightarrow{AD}$ take $B'$ so that (Theorem 4.6)

$$AB' = AD\frac{AC}{AE}.$$

Then $AB'/AD = AC/AE$, and by Theorem 9.6, $\overleftrightarrow{B'C} \parallel \overleftrightarrow{DE}$. But by the Parallel Postulate there is only one line through $C$ parallel to $\overleftrightarrow{DE}$, and so $B'$ must lie on $\overleftrightarrow{BC}$. Hence $B' = B$, the intersection of $\overleftrightarrow{BC}$ and $\overleftrightarrow{AD}$, and this proves the theorem.

From Theorems 9.6 and 9.7 there follows the standard development of the theory of similar triangles. One can also give a proof of the Pythagorean Theorem not involving the annoying manipulations of areas discussed above, as follows:

Let $\Delta\ ABC$ have a right angle at $A$. If $D$ is the foot of the perpendicular from $A$ to $\overleftrightarrow{BC}$ then $D$ lies between $B$ and $C$. (Proof?) Let $X$ be the point on $\overrightarrow{BA}$ such that $BX = BD$, and let the perpendicular to $\overleftrightarrow{AB}$ at $X$ intersect $\overrightarrow{BC}$ in $Y$. (Why must this perpendicular intersect $\overrightarrow{BC}$?) Then $\Delta\ XBY \cong \Delta\ DBA$, and so $BY = BA$. Also, $\overleftrightarrow{XY} \parallel \overleftrightarrow{AC}$ (Corollary 8.1), and by Theorem 9.7 $BX/BA = BY/BC$. This gives $BD/BA = BA/BC$, or

$$BA^2 = BC \cdot BD.$$

Similarly we can prove

$$CA^2 = CB \cdot CD.$$

Adding these two equations gives

$$BA^2 + CA^2 = BC(BD + CD) = BC^2.$$

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5. Alternate Proof of the Basic Theorems. In Chapter 2 we remarked that the basic theorem on proportionality can be proved independently of the area postulates, but that such a proof necessarily involves rather sophisticated manipulations of real numbers. For comparison with the fairly simple proof of Theorem 9.6 given above we present here the alternate proof.

Since the Theorem of Pythagorous follows from Theorem 9.7 we can thus obtain both theorems without using the area postulates.

Theorem 9.6. If B and D are distinct points on one side of $\angle A$, and C and E points on the other side, such that $AB/AD = AC/AE$, then $\overleftrightarrow{BC} \parallel \overleftrightarrow{DE}$.

Proof: As in the previous proof we shall consider the case in which $AB/AD < 1$. Let $\ell$ be the line through B parallel to $\overleftrightarrow{DE}$. By Theorem 5.4 applied to $\Delta$ ADE, $\ell$ intersects either $\overline{AE}$ or $\overline{DE}$ (or both). Being parallel to $\overleftrightarrow{DE}$ it cannot intersect $\overline{DE}$, and so $\ell$ intersects $\overline{AE}$ in a point F.

If $F = C$ then $\overleftrightarrow{BC} \parallel \overleftrightarrow{DE}$ and the theorem is proved. So suppose $F \neq C$; then $AF \neq AC$. Since we are given that

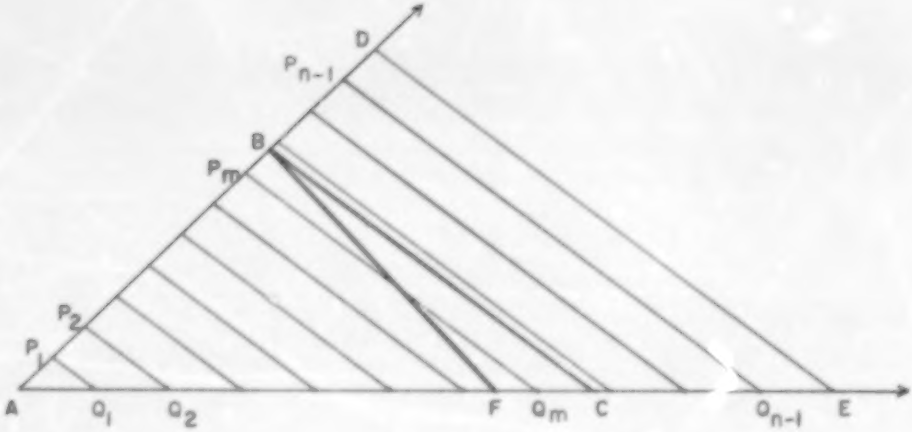
$$\frac{AB}{AD} = \frac{AC}{AE},$$

we must have

$$\frac{AB}{AD} \neq \frac{AF}{AE}.$$

Now one of the properties of the real number system is that between any two distinct numbers there is a rational number (a proof of this is given below). Hence there are positive integers m and n such that either

$$(*) \qquad \frac{AB}{AD} < \frac{m}{n} < \frac{AF}{AE} \quad \text{or} \quad \frac{AB}{AD} > \frac{m}{n} > \frac{AF}{AE}.$$



(The figure is drawn for the latter case.) On $\overrightarrow{AB}$ take points $P_1, P_2, \ldots, P_{n-1}$ such that

$$AP_1 = \frac{1}{n}AD,\ AP_2 = \frac{2}{n}AD,\ \ldots,\ AP_{n-1} = \frac{n-1}{n}AD.$$

Let the line through $P_1$ parallel to $\overleftrightarrow{DE}$ intersect $\overline{AE}$ in $Q_1$ (Theorem 5.4). Since $AP_1 = P_1P_2 = P_2P_3 = \ldots = P_{n-1}D$ we have from Theorem 8.9 that $AQ_1 = Q_1Q_2 = Q_2Q_3 = \ldots = Q_{n-1}E$.

Now AB/AD and AF/AE are each less than 1, and so from (*), $m < n$. Thus $P_m$ and $Q_m$ are two of our points constructed above. Hence

$$AP_m = mAP_1 = \frac{m}{n}AD,$$

$$AQ_m = mAQ_1 = \frac{m}{n}AE.$$

From these and (*) we get either

$$AP_m > AB \quad \text{and} \quad AQ_m < AF,$$

or

$$AP_m < AB \quad \text{and} \quad AQ_m > AF.$$

In either case it follows from Theorem 6.4a that $\overline{BF}$ and $\overline{P_mQ_m}$ intersect. This contradicts the assumption that $\overleftrightarrow{DF}$ and $\overleftrightarrow{P_mQ_m}$ are both parallel to $\overleftrightarrow{CE}$, and so we cannot have $F \neq B$. This proves the theorem.

To complete our analysis of this theorem we must still prove the property of real numbers that gave us the critical inequality (*). The proof, which is of the type considered in Chapter 4, will be broken into a series of three steps.

Lemma 1. Given any real number $z$, there is an integer $N$ such that $N > z$.

Proof: If the theorem is not true, then $z$ is an upper bound of the set of integers. By the Completeness Axiom, there is then a least upper bound, that is, a number $w$ such that

(i) $n \leq w$ for every integer $n$;

(ii) If $n \leq v$ for every integer $n$, then $w \leq v$.

It follows from (ii) that if $v < w$, then for some integer $n_1$ we must have $n_1 > v$. (This statement is just the contrapositive of (ii).) Now take $v = w - 1$. Then we have $n_1 > w - 1$, or $n_1 + 1 > w$. But then (i) does not hold for $n = n_1 + 1$, and so we arrive at a contradiction, thus proving the lemma.

Although we do not use it here, it is perhaps worth stating a corollary to this lemma, known as the Axiom of Archimedes.

Corollary*. (Axiom of Archimedes). If $a$ and $b$ are positive real numbers there is an integer $N$ such that $Na > b$.

Lemma 2. Given any positive real number $z$, there is an integer $M$ such that $M < z \leq M + 1$.

Proof: By Lemma 1 there is an integer $N$ such that $N > z$. Hence there are at most $N - 1$ positive integers which are less than $z$. If there are none, take $M = 0$; if there are some take $M$ to be the largest one. This choice of $M$ obviously has the required properties.

The lemma is still true if we do not restrict $z$ to be positive, but we do not need this more general case.

Lemma 3. If $0 < x < y$ there is a rational number $r = M/N$ such that $x < r < y$.

Proof: Let $n > \frac{1}{y - x}$, by Lemma 1, and let $M$, by Lemma 2, be such that

$$M < Ny \leq M + 1.$$

From the left inequality we get $r < y$. The right inequality is

$$Ny \leq M + 1$$

and from the choice of $N$ we have

$$Ny - Nx > 1.$$

Hence

$$Nx < Ny - 1 \leq (M + 1) - 1 = M,$$

or $x < \frac{M}{N} = r$. Thus $x < r < y$, as was to be proved.

There is still another approach to the basic proportionality theorem that avoids both the above difficulties and the use of the area postulates. This is to assume the statement of the theorem as a postulate. As was remarked in Chapter 2 the introduction of new postulates just to avoid difficult proofs is not to be encouraged. In this case, however, it turns out that by assuming as a postulate a statement closely related to Theorems 9.6 and 9.7 (the side-angle-side proposition for similar triangles) we can dispense with both the Congruence Postulate and the Parallel Postulate, being able to prove them from the new Similarity Postulate.

This is the approach taken by G. D. Birkhoff (see Birkhoff and Beatley, _Basic Geometry_) and by S. MacLane (_Metric Postulates for Plane Geometry_, American Mathematical Monthly, 66(1959)543-555). It has the advantage of elegance, in replacing two important postulates by one, but the pedagogic disadvantage of requiring the introduction of similarity before the simpler concepts of congruence and parallelism have become familiar. Also, it rules out the development of the non-Euclidean geometries. It is for these reasons that we follow the more conventional program.

## Chapter 10

## *Circles and Spheres*

### 1. *Basic Intersection Properties*.

*Definitions*. Given a point Q and a positive number r, the *sphere* with *center* Q and *radius* r is the set of all points whose distance from Q is r. If p is a plane containing Q, the *circle* in p with center Q and radius r is the set of all points in p whose distance from Q is r.

The following theorem is an immediate consequence of these definitions,

*Theorem 10.1*.* The intersection of a sphere with a plane containing its center is a circle with the same radius and center as the sphere.

Such a circle is said to be a *great circle* of the sphere.

We now consider the possible intersection of a sphere with a plane not containing its center.

Theorem 10.2. Let S be a sphere with center Q and radius r, let p be a plane not containing Q, let M be the foot of the perpendicular from Q to p, and let $a = QM$. Then

(1) If $a > r$, p and S do not intersect,

(2) If $a = r$, the intersection of p and S consists of the single point M.

(3) If $a < r$, the intersection of p and S is a circle with center M and radius $\sqrt{r^2 - a^2}$.

Proof: If $A$ is any point of $p$ then $\angle AMQ$ is a right angle, and so

$$AM^2 + MQ^2 = AQ^2.$$

If $A$ is also to be a point of $S$, then $AQ = r$. Since $MQ = a$, we get in this case

$$(*) \qquad AM^2 = r^2 - a^2.$$

(1) This is impossible if $a > r$, since $AM^2$ cannot be negative. Hence in this case there cannot be any such point as $A$; that is, $p$ and $S$ do not intersect,

(2) If $a = r$, $(*)$ says that $AM = 0$. This is true only if $A = M$; that is, $M$ is the only point common to $p$ and $S$.

(3) If $a < r$ then $a^2 - r^2$ is a positive number, which has a positive square root $b = \sqrt{a^2 - r^2}$. $(*)$ then says that $AM = b$; that is, $A$ can be any point in $p$ whose distance from $M$ is $b$. The set of such points is the circle in $p$ with center $M$ and radius $b$.

Note that Theorem 10.1 can be considered as a special case of Theorem 10.2 if we remove the restriction that $p$ not contain $Q$ and allow $a$ to be zero.

Definition. A plane containing just one point of a sphere is said to be tangent to the sphere at that point.

Definition. The segment joining the center of a sphere to any point of the sphere is called the radius of the sphere to that point.

As in Chapter 9, this double use of the word "radius" to mean either a segment or the length of that segment seldom causes confusion.

*Theorem* *10.3*$^*$. A plane tangent to a sphere is perpendicular to the radius at the point of contact, and conversely a plane perpendicular to a radius at its end-point on the sphere is tangent to the sphere at this point.

*Theorem* *10.4*$^*$. Let $C$ be a circle with center $Q$ and radius $r$, $\mathcal{L}$ a line in the plane of $C$, $M$ the foot of the perpendicular from $Q$ to $\mathcal{L}$, and $a = QM$. (We allow the possibility $M = Q$, $a = 0$.) Then

(1) If $a > r$, $\mathcal{L}$ and $C$ do not intersect,

(2) If $a = r$, $\mathcal{L}$ and $C$ intersect in the single point $M$.

(3) If $a < r$, the intersection of $\mathcal{L}$ and $C$ consists of exactly two points $A_1$ and $A_2$ such that $MA_1 = MA_2 = \sqrt{r^2 - a^2}$.

*Definition*$^*$. Line tangent to a circle.

*Theorem* *10.5*$^*$. Analogous to Theorem 10.3.

*Definitions*. A *chord* of a circle is a segment whose end-points lie on the circle. A chord which contains the center is a *diameter*. (The word "diameter" is also used as the length of this chord.) The *interior* of a circle is the set of all points in the plane of of the circle whose distance from the center is less than the radius of the circle; the *exterior* is the set of points whose distance is greater than the radius.

*Theorem* *10.5*$^*$. Every point of a chord except the end-points is in the interior of the circle.

*Theorem 10.6**. Every point of a tangent except the point of contact is in the exterior of the circle.

*Theorem 10.7**. Let $\overline{AB}$ be a chord of a circle with center $Q$ and let $M$ be the mid-point of $\overline{AB}$. of the following three properties of a line $\ell$:

(a) $\ell$ contains $Q$,

(b) $\ell$ contains $M$,

(c) $\ell$ is perpendicular to $\overleftrightarrow{AB}$;

if any two are true so is the third.

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2. *Arcs and Angles*. Throughout this section we shall consider a fixed circle $W$ with center $Q$ and radius $r$, and all figures will be assumed to lie in the plane of $W$.

*Definitions*. Let $A$ and $B$ be different points of $W$. If $\overline{AB}$ is not a diameter of $W$ the union of $A$, $B$, and all points of $W$ in the interior of $\angle AQB$ is called an *arc* $AB$; $A$ and $B$ are the *end-points* of the arc. Also the union of $A$, $B$, and all points of $W$ in the exterior of $\angle AQB$ is called arc $AB$, again with $A$ and $B$ as end-points. If one wishes to distinguish between these two, the former is called a *minor* arc, the latter a *major* arc.

If $\overline{AB}$ is a diameter of $W$, arc $AB$ is defined to be the union of $A$, $B$, and all points of $W$ lying on one side of $AB$. Such an arc is called a *semicircle*.

If arc $AB$ is a minor arc, $\angle AQB$ is called a *central angle*, and is said to intercept arc $AB$.

In some ways arcs behave like segments, and in some ways they behave like angles. The fact that we can have two arcs with the same end-points further complicates matters. If C is a point of W distinct from A and B we can distinguish the two arcs AB by the fact that one contains C and the other does not; the former is sometimes specified as arc ACB. The different kinds of arc, major, minor, and semicircle, require our basic proofs to consider several cases. All in all, a careful treatment of arcs is a tedious process, and not a suitable subject for a beginning geometry course. The following detailed proofs are therefore primarily for the benefit of the teacher.

*Definition.* With each arc AB there is associated a positive number, called the *measure* of the arc, denoted by $m(\text{arc } AB)$, defined as follows:

(1) If arc AB is minor, $m(\text{arc } AB) = m(\angle AQB)$,

(2) If arc AB is major, $m(\text{arc } AB) = 360 - m(\angle AQB)$,

(3) If arc AB is a semicircle, $m(\text{arc } AB) = 180$

The following theorem for arcs is analogous to Postulate 13 for angles. It is worded in a form which is suitable for proof and applications, though apt to be confusing at first reading. A figure will help to clarify the situation.

*Theorem 10.8.* Let A, B, C be different points on W. Let arc AB contain C, arc AC not contain B, and arc BC not contain A. Then

$$m(\text{arc } AB) = m(\text{arc } AC) + m(\text{arc } BC).$$

Proof: There are seven cases to be considered.

Case 1. Arc AB is a minor arc. Then C is in the interior of $\angle AQB$, and therefore B is in the exterior of $\angle AQC$ (Theorem 6.6), so that arc AC is minor. Similarly arc BC is minor. Hence each of the arcs has the same measure as its central angle. Since C is in the interior of $\angle AQB$,

$$m(\angle AQB) = m(\angle AQC) + m(\angle BQC),$$

and so

$$m(\text{arc } AB) = m(\text{arc } AC) + m(\text{arc } BC).$$

Case 2. Arc AB is a semicircle. The proof is essentially the same as for Case 1.

Case 3. Arc AB is major, and arc AC is major. Since B is not on the major arc AC, B is in the interior of $\angle AQC$, and so

$$m(\angle AQC) = m(\angle AQB) + m(\angle BQC).$$

A is in the exterior of $\angle BQC$, and since A is not on arc BC, arc BC must therefore be minor. Thus

$$\begin{aligned} m(\text{arc } AB) &= 360 - m(\angle AQB), \\ m(\text{arc } AC) &= 360 - m(\angle AQC), \\ m(\text{arc } BC) &= m(\angle BQC). \end{aligned}$$

From these equations and the one above we get again

$$m(\text{arc } AB) = m(\text{arc } AC) + m(\text{arc } BC).$$

Case 4. Arc AB is major, arc AC is a semicircle. Similar to Case 3.

Case 5. Arc AB is major, arc AC is minor, arc BC is minor. Here

$$m(\text{arc } AB) = 360 - m(\angle AQB),$$
$$m(\text{arc } AC) = m(\angle AQC),$$
$$m(\text{arc } BC) = m(\angle BQC).$$

If A and B were on the same side of $\overleftrightarrow{QC}$, then by Theorem 6.2 either A would be in the interior of $\angle BQC$ or B would be in the interior of $\angle AQC$. The first of these is ruled out because arc BC is a minor arc which does not contain A, the second similarly. Hence A and B are on opposite sides of $\overleftrightarrow{QC}$, and so $\overline{AB}$ intersects $\overleftrightarrow{QC}$. $\overline{AB}$ cannot intersect $\overrightarrow{QC}$ since C, being on the major arc AB, is in the exterior of $\angle AQB$. Hence $\overline{AB}$ intersects ray $\overrightarrow{QD}$ opposite to $\overline{QC}$, and so D is in the interior of $\angle AQB$. We then have

$$m(\angle AQB) = m(\angle AQD) + m(\angle BQD),$$
$$m(\angle AQC) + m(\angle AQD) = 180,$$
$$m(\angle BQC) + m(\angle BQD) = 180.$$

From these and the three previous equations we get

$$m(\text{arc } AB) = m(\text{arc } AC) + m(\text{arc } BC).$$

Case 6. Arc AB is major, arc AC is minor, arc BC is major. Same as Case 3 with A and B interchanged.

Case 7. Arc AB is major, arc AC is minor, arc BC is a semicircle. Same as Case 4 with A and B interchanged.

Definition. If A, B, P are different points of a circle, $\angle APB$ is said to be inscribed in the arc APB and to intercept the arc AB not containing P.

*Theorem 10.9*. The measure of an inscribed angle is half the measure of the intercepted arc.

*Proof*: Let $\angle APB$ be inscribed in $W$. We consider three cases.

Case 1. $\overline{PA}$ is a diameter. Since $P$ is in the exterior of $\angle AQB$ ($P$ and $A$ are on opposite sides of $\overleftrightarrow{QB}$) and $P$ is not on arc $AB$ it follows that arc $AB$ cannot consist of the points exterior to $\angle AQB$: That is, arc $AB$ is a minor arc, and $m(\text{arc } AB) = m(\angle AQB)$. Hence we have only to prove that $m(\angle APB) = \frac{1}{2}m(\angle AQB)$. Now in $\Delta QBP$, $QB = QP$, and so $m(\angle QBP) = m(\angle QPB)$ (Theorem 7.1). But $\angle AQB$ is an exterior angle of $\Delta QBP$, and so

$$m(\angle AQB) = m(\angle QBP) + m(\angle QPB) = 2m(\angle APB).$$

In case $\overline{PB}$ is a diameter we proceed similarly.

Suppose then that $C$, the other end-point of the diameter through $P$, is neither $A$ nor $B$.

Case 2. Suppose $A$ and $B$ are on the same side of $\overleftrightarrow{QP}$. Either $B$ is in the interior of $\angle AQC$ or $A$ is in the interior of $\angle BQC$. Suppose the former (the latter case can be treated similarly); then, by Theorem 6.6, $A$ is not on the minor arc $BC$ and $C$ is not on the minor arc $AB$. We can therefore apply Theorem 10.8 to get

$$m(\text{arc } AB) + m(\text{arc } BC) = m(\text{arc } AC).$$

Hence

$$\begin{aligned} m(\text{arc } AB) &= m(\text{arc } AC) - m(\text{arc } BC) \\ &= 2m(\angle QPA) - 2m(\angle QPB), \end{aligned}$$

by case 1 above. Now $\overrightarrow{QA}$ is in the interior of $\angle BQP$ (since

$\overrightarrow{QB}$ is not in the interior of $\angle AQB$) and so $\overrightarrow{QA}$ intersects $\overline{BP}$ in a point D. By Theorem 10.5, $QD < QA$, and so D, being between Q and A, is on $\overline{QA}$. Thus Q and A are on opposite sides of $\overleftrightarrow{BP}$, and therefore $\overrightarrow{PB}$ is in the interior of $\angle QPA$. Hence

$$\begin{aligned} m(\angle QPA) &= m(\angle QPB) + m(\angle APB); \\ m(\angle APB) &= m(\angle QPA) - m(\angle QPB) \\ &= \tfrac{1}{2}m(\text{arc } AB). \end{aligned}$$

from above.

Case 3. A and B are on opposite sides of $\overleftrightarrow{QP}$. Then A is not in the interior of $\angle CQB$, nor B in the interior of $\angle AQC$; and so if arc AC and arc BC are minor arcs, and if arc AB contains C, then the conditions of Theorem 10.8 hold and we have

$$\begin{aligned} m(\text{arc } AB) &= m(\text{arc } AC) + m(\text{arc } BC) \\ &= 2m(\angle APC) + 2m(\angle BPC). \end{aligned}$$

We have therefore only to prove that

$$m(\angle APC) + m(\angle BPC) = m(\angle APB);$$

this will be true if we show that C is in the interior of $\angle APB$. Now $\overline{AB}$ intersects line $\overleftrightarrow{PC}$ in a point D, and by Theorem 10.5, D is inside the circle. Hence D is on $\overline{CP}$ and so $\overrightarrow{PC} = \overrightarrow{PD}$. By Theorem 6.1, $\overrightarrow{PC}$ therefore is in the interior of $\angle APB$. This completes the proof of the theorem.

This theorem is the basis for a sequence of theorems relating arcs and angles in various positions, and for another sequence of theorems relating lengths of segments of chords, secants, tangents, etc. The usual proofs can be applied, but care must be taken, as in the proof above, to specify which arcs one is using.